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Interim Report 4

COUPLED MODE THEORY FOR ADVANCED MICROWAVE DEVICES

Prepared for:

AERONAUTICAL SYSTEMS DIVISION
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

CONTRACT AF 33(657)-8343

By: *M. C. Pease*

STANFORD RESEARCH INSTITUTE

MENLO PARK, CALIFORNIA

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Approved:

Philip Rice

PHILIP RICE, MANAGER PHYSICAL ELECTRONICS LABORATORY

J. Noe

JERRE D. NOE, DIRECTOR ENGINEERING SCIENCES DIVISION

Copy No. 16

ABSTRACT

This is the fourth interim report of a study whose purpose is to develop and apply a generalized theory of the effects of coupling the modes of propagation such as occur in distributed microwave devices.

In Sec. I, we review the entire work done so far in this study. This review is in descriptive, rather than mathematical, terms to indicate the purposes and interrelations of the various parts of the program.

In Sec. II, the analysis of the class of non-uniform systems that we have been studying is carried further. This is the class of systems such that the derivative of the system matrix is expressible as a commutator of the system matrix with another operator. This class includes many of the interesting non-uniform systems, and it is a class that is tractable for analysis.

This analysis has been here extended in several directions. In Sec. II-B, certain simpler subclasses are studied in detail. The 2 by 2 case with hermitian system matrix is discussed first, and then the general 2 by 2 case. The class of systems such that the system matrix can be regarded as embedded in a space that is a representation of the general rotation group is then studied. Two formalisms are used, one using the infinitesimal transformations of the rotation group, and the other the corresponding spin matrices. The results are equivalent, but the difference of form may be useful.

In Sec. II-C we consider the general solution of the equation

$$\frac{d\mathbf{R}}{dz} = j[\mathbf{R}, \mathbf{A}]$$

for $\mathbf{A}$ in terms of $\mathbf{R}$ and $(d\mathbf{R}/dz)$. The basic mathematical problem, expressed in abstract and general terms, is given in the two appendixes in which we consider, first, when and how the equation

$$\mathbf{u} = \mathbf{T}\mathbf{v}$$

can be solved for $\mathbf{v}$, when $\mathbf{T}$ is a singular operator, and then some of the general properties of the commutator operator. These results are used to solve the system equation for the operator $\mathbf{A}$ under very general conditions. Thus we have solved the problem of actually finding the appropriate commutator operator for a non-uniform system of this type.

In Sec. II-D, the solution of such systems is considered. We show that it is possible to put such systems into a canonical form in which the simplicities of the system are exploited. The proceedure is in general to find a sequence of rotating axes in terms of which the system non-uniformity is progressively eliminated. The equations that determine these axes are given explicitly.

This work is in general, abstract and formal, but has been necessary to provide a foundation for the study of the practical possibilities of mode coupling by appropriately designed non-uniformity.

FOREWORD

This program was established by the Aeronautical Systems Division, Wright-Patterson Air Force Base, Ohio, at Stanford Research Institute under Contract AF 33(657)-8343 for the purpose of advancing the general understanding of microwave devices through the development and application of a general theory of coupling of modes of propagation.

At Stanford Research Institute the project supervisor is Philip Rice, and the principal investigator is Marshall C. Pease.

This is ASD Project 4165; Task 41651; the project monitor is Lawrence F. Daum (ASRNET-3) of the Electronic Technology Laboratory (ASRNE).

CONTENTS

I SUMMARY OF WORK

In this section we shall review the work of the contract to date, attempting to put the various pieces of reported results into proper relation with each other.

The purpose of the contract, as stated in the contract, is to "advance our engineering design knowledge and theoretical understanding of electron beam microwave devices through the derivation and use of the theory of coupling between the modes of propagation existing in a beam-type device." As stated in the original proposal, the broad objective was "to provide the analytic structure whereby existing design concepts can be interrelated and evaluated and new concepts generated." We also stated that our general approach would be an attempt to "exploit some of the more advanced techniques of matrix algebra and abstract vector spaces."

In carrying out this objective, our work has been directed towards two broad classes of structures—the sectionally uniform and the non-uniform types.

By a "uniform" system or section we mean one whose design parameters do not change along the length. The electron beam velocity does not change along the distance. The impedance and propagation constant of the transmission line are constant. The magnetic field, if one is present, is the same at all points being considered. And so on. A non-uniform system or section is one in which some one or more of these parameters varies along the axis.

Most existing devices either are "sectionally uniform" or can be treated as if they were. A travelling wave tube may be considered as being built of sections which are uniform. Even if the tube is focused by a periodic magnetic field, the periodicity is carefully chosen so that the results will approximate the behavior the tube would have if a single solenoidal field was used.

One exception is the parametric amplifier with travelling pump signal. The pump signal in effect imposes a time dependence, at the pump frequency, of certain of the parameters of the system. Further, if the pump wave is travelling, there is a variation of phase along the interaction space. This, then is a "non-uniform" system.

There is, furthermore, strong indication that a properly chosen non-uniformity can lead to new and potentially interesting effects.

There is, for example, experimental evidence suggesting that variation, with z, of the magnetic or electric field under crossed field conditions may provide strong amplification. If so, this might permit strong amplification with a very simple physical structure. Such an effect might be useful for several purposes. It is worth considering, for example, if there is need of compact amplifiers for use at low frequencies. It would be a good possibility for millimeter or submillimeter wave amplification, except that in this region it implies an average magnetic field strength that is probably impractical. However, the thought does illustrate the potentialities that exist of using a controlled non-uniformity to achieve a desired mode coupling.

The analysis of the effects of non-uniformity is still in a very unsatisfactory state. Certain situations of particular interest—*e.g.*, periodic focusing and parametric interaction—have been analyzed by special methods. But no general theory exists. There is no theory that will tell us what is possible in such systems, or that will give us any indications at all of how the possible can be achieved.

Thus, there appears to be, here, a large area of possible design that has not been exploited to any significant degree.

It is then in these two areas that we have concentrated our attention.

A. SECTIONALLY UNIFORM DEVICES

The technique for the analysis of the mode couplings that can occur in structures that are sectionally uniform, or that can be so approximated, was in fairly good shape at the start of the program. Methods of analysis that are always applicable, at least in principle, were available. The principal phenomena that are possible in such systems were known. And a good deal was known, for the simpler cases at least, about the conditions under which a given phenomenon such as amplification or power transfer will occur.

Although the principal phenomena that are possible and the conditions for their occurrence were known, the material was widely scattered in the literature and in our work under a previous contract [AF33(616)-5803]. Nowhere, it seemed, had it been brought together and codified into a single coherent theory.

One of the specific objectives of this contract as indicated in the contract itself, was to correct this deficiency. Therefore, during the contract we prepared and issued Technical Note 1, "Topological Analysis of Sectionally Uniform and Parametric Systems." This covers in a qualitative and semi-quantitative way the phenomena that are possible as a result of the uniform coupling of modes by pairs. In so doing, we have provided a method for diagraming most existing devices in a way that exhibits the principal local effects and allows one to deduce the nature of the principal over-all consequences.

The report was difficult to write because of the danger of swamping the reader in a mass of definitions and mathematical details. The precise definition of terms and the essential mathematical structure could not be avoided without making the report an essentially superficial one. But neither could they be allowed to dominate the report without making the report useful only to the expert in the field—who presumably knows the material anyway. The report was, therefore, written with great care to present the essential features of the theory first, filling in the details later and in the Appendixes. It is hoped that, by this means, it has been put into a form that will be useful.

The Technical Note was primarily concerned with the pairwise coupling of modes. The simultaneous coupling of a larger group of modes was avoided, except for the inclusion of what we call "chain coupling," in Appendix C. This is a more general type of coupling that is, mathematically, closely related to pairwise coupling. (Pairwise coupling can be regarded as the chain coupling of a chain of length two.) It was included in the Technical Note because its mathematical development includes, without being much more complicated than, the necessary mathematical development for simple pairwise coupling. Also, it includes situations that may exist as disrupting phenomena in present devices.

The theory of chain coupling was developed and reported originally in Interim Report 2, Secs. II-B and II-C. The version of this theory in Appendix C of the Technical Note is a rewrite of these, and contains essentially the same material.

The Technical Note also included the theory and effect of pairwise parametric coupling. This is a non-uniform situation, but one which can be handled in a manner that is closely related to the procedures for uniform systems. Also, of course, parametric coupling is used in a very important class of existing devices. Since it was the intention in the Technical Note to bring together, insofar as possible, all the coupled mode theory necessary to the qualitative understanding of all existing linear electron beam devices, it was appropriate that parametric theory should be included.

A different approach to the study of uniform systems was developed and reported in Interim Report 1, Sec. II-A. This used what we called the "System Function Characteristic." This procedure is somewhat analogous to the transfer function used in network theory for calculation transient responses. It appears to be particularly useful in the analysis of parametric situations in which an indefinitely large number of frequencies may be involved.

Application of this method was made to a device in which two pump signals are imposed simultaneously. Under suitable conditions, this will couple the original signal to successively higher parametric harmonics as the process continues.

We have suggested that this might be a useful device for the generation of high frequencies by harmonic interaction. A final evaluation of the possibilities here has not yet been made, but the prospects do look interesting.

Finally, we should note one other area of interest that has not yet been tackled at all. This is the case where statistical methods are required. This may happen because of the very large number of modes that may be involved, such as in the "laser." Or it may happen because of the statistical nature of the processes that are involved. There is, for example, the possibility of using noise to pump a parametric amplifier.

There is need for techniques to handle these problems. No effort has as yet been made in this direction, although the group theoretic methods to be discussed shortly may provide a useful starting point. However, specific attention should be given to this class of problems when time permits.

B. NON-UNIFORM SYSTEMS

The other major area that has concerned us is the analysis of non-uniform systems. No general understanding exists of the possibilities of such systems. Only certain rather narrowly restricted types of non-uniform systems, such as those using conventional parametric interaction, have been analyzed. And yet there is evidence that non-uniformity can provide the means for obtaining some badly needed design capabilities such as strong amplification without excessively intricate structure.

We have made substantial progress on this problem although it must be admitted that the results are so far quite fragmented and have not yet been coordinated into a single, useful body of theory.

The earliest approach was by an adaptation of the methods of classical physics. This was reported in Interim Report 1, Sec. II-B. "A Formal Lagrangian Function and Related Concepts." It was shown there that it was possible to define a Lagrangian Function and a Hamiltonian, and to obtain from them such related concepts as the Poisson Bracket and the canonical equations of motion. It was also found possible to define appropriate contact or canonical transformations, and to classify systems according to their infinitesmal contact transformations. Although these definitions are formal—in the sense that the Hamiltonian, for example, has little if any relation to the total energy of the system—this approach appears to open the way to the employment of the various methods that have proven so powerful in classical and quantum mechanics.

We have not yet carried this approach further, mainly because of the pressure of other ideas. It is, however, a technique that we continue to regard hopefully, and that we intend to return to at some future time.

Another approach, which was partly generated out of the preceding one, is the use of group theoretic methods. This was reported in

Interim Report 3, Sec. I. It arose out of the formal methods since the infinitesmal contact transformations of a system do form a group that in an appropriate sense describes the system.

The group theoretic description of a system is primarily useful for the purposes of classification and representation. At least this has been the principal use of this approach so far. It has proven to be a powerful method of obtaining a convenient representation for certain types of non-uniform systems. As such, it provides the means for the further analysis of such systems.

It can also be added that these methods are likely to be highly important when we come to develop a statistical theory of systems with large numbers of modes, such as "lasers." In such systems, the principal information we are likely to have regarding existing couplings will derive from symmetry considerations—that is, from group theoretic properties. It seems likely, therefore, that the problem can be made most tractable by treating it in at least a group theoretic type of representation.

The third approach, which seems to be in the process of yielding important results, concerns the analysis of a special class of non-uniform systems. This class includes parametric systems, the exponentially tapered transmission line, and most of the systems which have been individually solved to date. It also includes much more complicated systems. It is a class, however, that has some very important properties that make its analysis tractable.

This class of systems is important, not only for the specific devices that it includes, but also as an analyzable class of considerable variety. Thus, we can use it to explore what is possible in non-uniform systems and to determine conditions under which a given type of behavior can be obtained.

The class referred to is described mathematically as that in which the derivative of the system operator is expressible as a commutator of the operator with another operator (*cf.* Interim Report 2, Sec. II-A). It may also be said to be the class in which the behavior of the system is essentially rotational, although this statement would require considerable amplification in terms of the geometry of the vector space involved, and the meaning of "rotation" in this connection. This viewpoint is at least partly developed and exploited in Sec. II-B below.

The first results and the beginning of recognition of the full class were reported in Interim Report 2, Sec. II-A. This was carried on, and the full class defined, in Interim Report 3, Sec. II. Section II of the present report continues various aspects of this analysis.

This analysis is still somewhat fragmented in that we are not yet ready to pull together the pieces into a coherent body of theory. It is also still quite incomplete in that the work so far has been dealing with various aspects of the underlying theory, with only minor attention being given to practical results. This has been necessary because of the apparently almost complete lack of previous work on this class of problems. Hopefully, however, we are now about ready to study directly the practical problems that are the reason for our interest in this field.

It is dangerous to predict progress in an area about which so little is known as this one. However, it does appear that there is reason to be hopeful that we are now in a position to make substantial progress in the understanding of how non-uniformity can be used for practical purposes.

II ANALYSIS OF NON-UNIFORM SYSTEMS

A. INTRODUCTION AND SUMMARY

We shall, in the various sections that follow, continue our analysis of non-uniform systems of the restricted class we have considered in Sec. II-A of Interim Report 2 and Sec. II of Interim Report 3. A non-uniform system is one which is described by a vector differential equation of the form:

$$\frac{d\mathbf{x}(z)}{dz} = -j\mathbf{R}(z)\mathbf{x}(z) \tag{1}$$

where $\mathbf{x}$ is the "state vector" represented as a column matrix describing the state of the system at the position z, and $\mathbf{R}$ is the system matrix. The system is non-uniform if $\mathbf{R}$ is dependent on z.[1]

The particular class of systems that we will consider are those for which $\mathbf{R}(z)$, although not constant, is such that its derivative is expressible as the commutator of $\mathbf{R}$ with some matrix $\mathbf{A}$:

$$\frac{d\mathbf{R}}{dz} = j[\mathbf{R}, \mathbf{A}] = j(\mathbf{RA} - \mathbf{AR}) \quad . \tag{2}$$

Our interest in this class arises from the fact, discussed in Sec. II of Interim Report 3, that certain subclasses of this class are both practically important and explicitly soluble. Thus, this class may be a useful one for exploring what can be done with non-uniformity in physical systems. Since use is made of non-uniformity in some important systems such as distributed parametric amplifiers, it is not unreasonable to hope that the study of such systems may open the way to important new devices.

[1] As usual, we shall indicate column vectors by bold faced lower case letters, and matrices by bold faced capital letters. We shall indicate the complex conjugate by (*), and the hermitian conjugate, or complex conjugate transpose, by (†).

The matrix, $\mathbf{A}$, is not well defined by Eq. (2). We can add to $\mathbf{A}$ any matrix that commutes with $\mathbf{R}$, without affecting $\mathbf{R}$ in Eq. (2). In particular, we can add to $\mathbf{A}$ any function of $\mathbf{R}$ and z, $f(\mathbf{R}, z)$, which is well defined [specifically, $f(u, z)$ is a suitable generating function if for any value of z, $f(u, z)$, considered as a function of u, is analytic in a simply connected region that includes the eigenvalues of $\mathbf{R}$ at that value of z].

We showed, in Sec. II-A of Interim Report 2, that a necessary and sufficient condition for the existence of such an $\mathbf{A}$ is that the eigenvalues and structure of $\mathbf{R}$ be constant. This does not, however, indicate whether or not $\mathbf{A}$ can be chosen to be constant. If, however, an $\mathbf{A}$ can be found which is constant, then Eq. (1) is explicitly soluble.

In Sec. II of Interim Report 3, we showed that this process is generalizable. In particular, if no constant $\mathbf{A}$ can be found for a given $\mathbf{R}$, we may still be able to find an $\mathbf{A}$ that has constant eigenvalues and structure so that its derivative is expressible as the commutator with a matrix $\mathbf{B}$:

$$\frac{d\mathbf{A}}{dz} = j[\mathbf{A}, \mathbf{B}] \quad . \tag{3}$$

If a constant $\mathbf{B}$ can be found, then Eq. (1) is still explicitly soluble. If not, but if $\mathbf{B}$ can be chosen so that is obeys the same type of equation,

$$\frac{d\mathbf{B}}{dz} = j[\mathbf{B}, \mathbf{C}] \tag{4}$$

and if $\mathbf{C}$ can be chosen to be constant, then again an explicit solution to Eq. (1) is obtainable, although one that involves an infinite series.

Thus this class of systems does include some soluble subclasses of considerable variety, and the whole class appears to have some interesting and useful simplicities, compared to the whole class of non-uniform systems.

The immediate need in the analysis of this class is for means of determining an $\mathbf{A}$ from the properties of $\mathbf{R}$ and its derivatives to provide a tool for the determination of the simplest $\mathbf{A}$—constant, if possible, otherwise expandable according to Eq. (3).

An expression for an **A** was given in Sec. II-A of Interim Report 2 in terms of the matrix **S**(z) which reduces **R** to canonical form—*i.e.*, the matrix of eigenvectors of **R**. Another expression was given in Sec. II of Interim Report 3, expressing **A** in terms of the dyads formed from the eigenvectors and generalized eigenvectors of **R** and their derivatives.

Neither of these expressions is entirely satisfactory for deeper analysis. Both depend explicitly on the eigenvectors (and, if necessary, generalized eigenvectors). The eigenvectors, however, may be multiplied by arbitrary scalar functions of z without disturbing the eigenvector properties. It is this fact that leads to the arbitrariness of **A**. However, there does not appear to be any simple relation between a z-dependent renormalization of the eigenvectors and the resultant change of **A**. It is not clear, therefore, how to specify the normalization of the eigenvectors so as to obtain an **A** matrix showing any desired properties.

In an effort to overcome this difficulty, we have tackled the problem again of determining **A** in Eq. (2) given **R**. This time we have sought a specification of **A**, or of a possible **A**, in terms only of **R** and its derivative.

In Sec. II-B of the present report, this problem is solved for various restricted classes of **R**, of increasing complexity. We are able, for these classes, not only to obtain explicitly a possible **A**, but have gone further and have been able to find **A**'s that are themselves expandable in the form of Eq. (3). Thus, for these classes, we have an explicit procedure for developing a chain of equations of the form of Eqs. (2), (3), (4), etc.

This solution has an importance beyond the purely theoretic. It includes the case when the non-uniformity introduces a pairwise coupling of the modes of the uniform system. It includes, then, phenomena of considerable practical importance. In preparation for the detailed investigation of these possibilities, we have, therefore, developed the explicit solutions in different but equivalent representations that appear to be of possible interest.

In Sec. II-C of the present report we consider the more general class of systems, restricted only by the condition that **R** is of simple structure. In this case, we are able to obtain an explicit solution for **A**. This does not complete the work needed, since the **A** so found is not, in general, expandable in the form of Eq. (3). We have not as yet been able to find a general expression for an expandable **A**. Indeed, we have not yet proven that an expandable **A** always exists, although we have gone far enough to feel fairly confident of the truth of this conjecture. Nevertheless, the development of an explicit expression for an **A**, given **R** and **R**$'$ and involving only the eigenvalues and not the eigenvectors of **R** does appear to be a substantial step forward.

In the course of developing this expression for an **A**, we have been led to investigate the "inversion" of a general singular operator. Quotation marks are used here, because a singular operator, by definition, has no inverse. Within the restricted context in which the problem arises, however, there does exist an operator that has the properties of an inverse. It is in this sense that we have investigated the problem using the techniques of abstract finitely dimensioned linear algebras. The work is here given in Appendix A.

This work also leads to a more general solution of Eq. (2) for **A**. Specifically, it permits us to remove the restriction, in Sec. II-C, that **R** be of simple structure. Since this restriction is of little practical importance, the development of this generalization is relegated to Appendix B. However, it may be noted that the resultant expression for **A**, even when **R** is of simple structure so that Sec. II-C applies, is not the same. (**A**, it will be recalled, has some arbitrariness to it.) It is possible that this more general solution may be of direct interest even where its generality is not important. This is speculation at this point, but the possibility should be kept in mind.

In Sec. II-D, we have undertaken the formalization and systemization of the process of solution of a chain of equations of the form of Eqs. (2), (3), (4), etc. The method of solution is that used in Interim Report 3, Sec. II-D. We find that we are able to interpret the process as a succession of z-dependent changes of basis. The transformation operators that accomplish this are chosen to obey a set of equations that we will call the canonical transformation equations of the chain.

B. NON-UNIFORM COUPLING BY PAIRS OF MODES

We have, in Interim Reports 2 and 3, considered non-uniform systems such that

$$\frac{d\mathbf{x}(z)}{dz} = -j\mathbf{R}(z)\mathbf{x}(z) \tag{5}$$

and

$$\frac{d\mathbf{R}}{dz} = -j(\mathbf{AR} - \mathbf{RA}) = -j[\mathbf{A}, \mathbf{R}] \tag{6}$$

$$\frac{d\mathbf{A}}{dz} = -j[\mathbf{B}, \mathbf{A}] \tag{7}$$

$$\frac{d\mathbf{B}}{dz} = -j[\mathbf{C}, \mathbf{B}] \tag{8}$$

etc.,

where the sequence terminates when and if a constant matrix is reached. We have been interested in such systems primarily because they are much simpler than the general non-uniform systems, and yet evidently retain much of the potentailities of non-uniformity.

If $\mathbf{A}$ can be chosen to be constant, then the system is a generalization of the exponentially tapered transmission line, and is directly soluble. If $\mathbf{A}$ cannot be taken as constant, but $\mathbf{B}$ can be, then the system is still explicitly soluble in closed form. If $\mathbf{C}$ is constant and the others not, then it is still explicitly soluble, although the solution involves an infinite sum. Thus, while the complexity increases rapidly, as we increase the length of thesequence of commutator relations, Eqs. (5), (6), (7), (8), etc., the problem still retains a degree of tractability.

The practical importance of these systems lies in the fact that they include systems in which mode coupling is induced by an appropriate non-uniformity of the system parameters.

We shall, in this section, analyze in detail systems of this type in which only a pair of modes is involved. This is probably the most interesting case from the practical point of view. The generalization of the analysis to more complex situations will be deferred until later.

We shall first discuss the situation when $\mathbf{R}$ is a 2×2 hermitian matrix. This is the simplest situation. Historically, its analysis provided the concepts that opened the way to the more general case. It is hoped that its presentation here may provide similar insight to others.

We shall follow this, then, with an analysis, first, of the situation with a general 2×2 matrix, and then of the situation when **R** is isomorphic to the rotation group. In the final case, we achieve a considerable degree of generality, although still not the most general case we would like to be able to handle.

1. **K** = **I**, **R** Hermitian, $n = 2$

We consider the case when $\mathbf{R}(z)$ is restricted to be a 2×2 matrix that is hermitian at all z. In our usual terminology, **R** is K-hermitian, with **K** = **I**, the identity matrix.

We have proven, that, for Eq. (6) to hold at all, it is necessary and sufficient that the eigenvalues of **R** be constant, with the z-dependence of **R** involving only a change of direction of its eigenvectors.

We have also shown that if **R** is K-hermitian, we can so choose **A** that it is everywhere K-hermitian. The matrix **A** is not completely determined by Eq. (6). To a given **A**, we can add any matrix function of z that commutes with **R**, without changing Eq. (6). It is for this reason that we can only say that **A** *can* be chosen to be everywhere K-hermitian, not that it must be.

If an **A** exists—*i.e.*, **R** has constant eigenvalues—then the first question we ask is, does a constant **A** exist? If so, and if we can find it, the problem is solved. If a constant **A** does not exist, then our next question is, does an **A** with constant eigenvalues exist? If so, we can write Eq. (7) and begin all over to examine the properties of **B**. If it is impossible to find an **A** that is either constant, or has constant eigenvalues, then the whole procedure fails and we must seek a solution by other means.

The central problem of this mode of analysis is the question of whether or not it is always possible to so choose **A** that *its* eigenvalues are constant, so that we can then find a **B** that will fit Eq. (7). It is with this problem that we are mainly concerned here.

If we write **R** as

$$\mathbf{R} = \begin{pmatrix} \theta & \psi \\ \xi & \phi \end{pmatrix} \tag{9}$$

then the requirement that **R** be hermitian specifies that θ and ϕ be real functions of z, and that ξ be the complex conjugate of ψ.

The characteristic equation of Eq. (9) is

$$\lambda^2 - (\theta + \phi)\lambda + (\theta\phi - \psi\xi) = 0 \quad . \tag{10}$$

If the roots of Eq. (10), the eigenvalues of **R**, are constant, then Eq. (10) must be constant, or the trace and the determinant of **R** must be constant. Hence Eq. (9) can be written as

$$\mathbf{R} = \begin{pmatrix} m + f & g + jh \\ g - jh & m - f \end{pmatrix} \tag{11}$$

where m is a real constant, and f, g, and h are real functions of z with the supplementary condition that

$$f^2 + g^2 + h^2 = a^2 \tag{12}$$

a being constant.

The matrix **R** of Eq. (11) with the constraint of Eq. (12) is the most general form of hermitian **R** with constant eigenvalues and which therefore is expressible as Eq. (6).

We can look for an **A** of the same form as Eq. (11).

$$\mathbf{A} = \begin{pmatrix} M + F & G + jH \\ G - jH & M - F \end{pmatrix} \tag{13}$$

where the constraint that

$$F^2 + G^2 + H^2 = b^2 \tag{14}$$

will be applied later.

If we substitute Eqs. (11) and (13) in Eq. (6), we find that we require

$$\begin{aligned} f' &= 2(hG - gH) \\ g' &= 2(fH - hF) \\ h' &= 2(gF - fG) \end{aligned} \tag{15}$$

where the prime means (d/dz).

We may note that m and M do not enter into these equations or into Eqs. (12) and (14). The constant m is part of $\mathbf{R}$, but does not affect the choice of $\mathbf{A}$. The constant M can be chosen arbitrarily and might as well be taken as zero.

If, now, we define the 3-vectors:

$$\begin{aligned} \vec{u} &= \vec{i}f + \vec{j}g + \vec{k}h \\ \vec{v} &= \vec{i}F + \vec{j}G + \vec{k}H \end{aligned} \tag{16}$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are the three unit vectors of a cartesian coordinate system, then Eq. (15) can be written as

$$\vec{u}\,' = -2\,\vec{u} \times \vec{v} \quad . \tag{17}$$

In other words, the right side of Eq. (15) is a cross product relation.

Equations (12) and (14) can, then, be written as

$$\vec{u} \cdot \vec{u} = a^2 \tag{18}$$

$$\vec{v} \cdot \vec{v} = b^2 \quad . \tag{19}$$

Equation (18) is obtainable from Eq. (17), and hence adds nothing new.

Equation (17) states that $\vec{u}'$ is perpendicular to $\vec{u}$ and $\vec{v}$. The vector $\vec{v}$, then, must have a component that is perpendicular to $\vec{u}$, but may have a parallel component. It is perpendicular to $\vec{u}'$. It is reasonable, therefore, to consider a solution for $\vec{v}$ of the form

$$\vec{v} = \alpha\vec{u} \times \vec{u}' + p\vec{u} \tag{20}$$

where α and p are scalar constants or functions to be determined.

If we substitute Eq. (20) and in Eq. (17) we see that

$$\begin{aligned} \vec{u}' &= -2\alpha\vec{u} \times (\vec{u} \times \vec{u}') \\ &= -2\alpha\{(\vec{u} \cdot \vec{u}')\vec{u} - (\vec{u} \cdot \vec{u})\vec{u}'\} \\ &= +2\alpha(\vec{u} \cdot \vec{u})\vec{u}' \\ &= +\alpha a^2\vec{u}' \quad . \end{aligned} \tag{21}$$

Hence Eq. (17) is satisfied if α is constant and

$$\alpha = \frac{1}{2a^2} \quad . \tag{22}$$

Substituting Eq. (20) in Eq. (19), we find

$$\begin{aligned} b^2 &= \alpha^2(\vec{u} \times \vec{u}') \cdot (\vec{u} \times \vec{u}') + 2\alpha p(\vec{u} \times \vec{u}') \cdot \vec{u} + p^2\vec{u} \cdot \vec{u} \\ &= \alpha^2\{(\vec{u} \cdot \vec{u})(\vec{u}' \cdot \vec{u}') - (\vec{u} \cdot \vec{u}')(\vec{u} \cdot \vec{u}')\} + a^2p^2 \\ &= \frac{1}{4a^2}(\vec{u}' \cdot \vec{u}') + a^2p^2 \end{aligned} \tag{23}$$

or

$$p^2 = (b^2/a^2) - (\vec{u}' \cdot \vec{u}')/(4a^4) \tag{24}$$

and p is determined as a function of z. The constant b must, of course, be taken as sufficiently large so that p^2 is nowhere negative if $\mathbf{A}$ is to be hermitian.

Hence, it is possible to determine a matrix $\mathbf{A}$ that satisfies Eq. (6) and that has constant eigenvalues so that it is expandable as Eq. (7).

We shall not pursue this further, here. Our main objectives were to find a method that will allow us to handle the more general problem, and to gain insight into the significance of Eq. (6) itself. We have, hopefully, accomplished these objectives in discovering that we are dealing with an operator that can be considered as the cross product of an entity— the vector $\vec{u}$—time the operand. This suggests that the operator $j[-, \mathbf{R}]$ be considered as, in some sense, the cross product of $\mathbf{R}$ and the operand. It suggests, then, that we consider whether this thought will allow us to tackle the more general case.

2. The General 2 × 2 R Matrix

We consider, now, the case where $\mathbf{R}$ is a 2×2 matrix function of z. Then the Cayley-Hamilton Theorem states that there must exist an a and b such that

$$\mathbf{R}^2 = a\mathbf{R} + b\mathbf{I} \quad . \tag{25}$$

Furthmore, a and b must be constants if the eigenvalues are to be constant.

If, now, we regard the operator $j[-,\mathbf{R}]$ as analogous to the operator $(\vec{u} \times -)$, then Eq. (6) becomes analogous to Eq. (17). The solution to Eq. (17) was found by considering a form expressible as Eq. (20). The analogous form would then be

$$\mathbf{A} = -j\alpha[\mathbf{R}', \mathbf{R}] + p\mathbf{R} + q\mathbf{I} \quad . \tag{26}$$

For this to satisfy Eq. (6), we must have

$$\mathbf{R}' = -\alpha(\mathbf{R}'\mathbf{R}^2 - 2\mathbf{R}\mathbf{R}'\mathbf{R} + \mathbf{R}^2\mathbf{R}') \tag{27}$$

Now using Eq. (25) and its derivative

$$\mathbf{R}\mathbf{R}' + \mathbf{R}'\mathbf{R} = a\mathbf{R}' \tag{28}$$

we find that

$$\begin{aligned}
\mathbf{R}'\mathbf{R}^2 \; -2\mathbf{R}\mathbf{R}'\mathbf{R} \quad & \mathbf{R}^2\mathbf{R}' \\
&= \mathbf{R}'\mathbf{R}^2 + \mathbf{R}^2\mathbf{R}' - 2\mathbf{R}(a\mathbf{R}' - \mathbf{R}\mathbf{R}') \\
&= \mathbf{R}'\mathbf{R}^2 + 3\mathbf{R}^2\mathbf{R}' - 2a\mathbf{R}\mathbf{R}' \\
&= \mathbf{R}'(a\mathbf{R} + b\mathbf{I}) + 3(a\mathbf{R} + b\mathbf{I})\mathbf{R}' - 2a\mathbf{R}\mathbf{R}' \\
&= a(\mathbf{R}\mathbf{R}' + \mathbf{R}'\mathbf{R}) + 4b\mathbf{R}' \\
&= (a^2 + 4b)\mathbf{R}' \quad .
\end{aligned} \tag{29}$$

This assumes that $(a^2 + 4b)$ does not vanish. There are cases in which it does vanish, so that Eq. (26) fails. These, however, are limiting cases and are, furthermore, soluble by other methods. In particular, it appears that, in these cases, $\mathbf{R}$ always commutes with its integral, so that

$$\mathbf{x}(z) = [\exp -j\int\mathbf{R}(z)dz]\mathbf{x}(0)$$

is the solution of Eq. (5)

We will assume that $a^2 + 4b$ does not vanish. If, then, we set

$$\alpha = 1/(a^2 + 4b) \tag{30}$$

we see that Eq. (26) satisfies Eq. (6). We note that α is necessarily constant.

Now, from Eq. (26), we have that

$$\begin{aligned}
\mathbf{A}^2 = & -\alpha^2[\mathbf{R}', \mathbf{R}]^2 - j\alpha p\{[\mathbf{R}', \mathbf{R}]\mathbf{R} + \mathbf{R}[\mathbf{R}', \mathbf{R}]\} \\
& - 2j\alpha q[\mathbf{R}', \mathbf{R}] + (p\mathbf{R} + qI)^2 \quad .
\end{aligned} \tag{31}$$

Considering these terms in turn, with the aid of Eqs. (25) and (28), we find that

$$\begin{aligned}
[\mathbf{R}'\mathbf{R}]^2 &= \mathbf{R}'\mathbf{R}\mathbf{R}'\mathbf{R} - \mathbf{R}\mathbf{R}'^2\mathbf{R} - \mathbf{R}'\mathbf{R}^2\mathbf{R}' + \mathbf{R}\mathbf{R}'\mathbf{R}\mathbf{R}' \\
&= \mathbf{R}'\mathbf{R}(a\mathbf{R}' - \mathbf{R}\mathbf{R}') - (a\mathbf{R}' - \mathbf{R}'\mathbf{R})(a\mathbf{R}' - \mathbf{R}\mathbf{R}') \\
&\quad - \mathbf{R}'\mathbf{R}^2\mathbf{R}' + (a\mathbf{R}' - \mathbf{R}'\mathbf{R})\mathbf{R}\mathbf{R}' \\
&= -4\mathbf{R}'\mathbf{R}^2\mathbf{R}' + 4a\mathbf{R}'\mathbf{R}\mathbf{R}' - a^2\mathbf{R}'^2 \\
&= -4\mathbf{R}'(a\mathbf{R} + b\mathbf{I})\mathbf{R}' + 4a\mathbf{R}'\mathbf{R}\mathbf{R}' - a^2\mathbf{R}'^2 \\
&= -(a^2 + 4b)\mathbf{R}'^2 \\
&= (1/\alpha)\mathbf{R}'^2
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
\{[\mathbf{R}', \mathbf{R}]\mathbf{R} + \mathbf{R}[\mathbf{R}', \mathbf{R}]\} &= \mathbf{R}'\mathbf{R}^2 - \mathbf{R}^2\mathbf{R}' \\
&= \mathbf{R}'(a\mathbf{R} + b\mathbf{I}) - (a\mathbf{R} + b\mathbf{I})\mathbf{R}' \\
&= a(\mathbf{R}'\mathbf{R} - \mathbf{R}\mathbf{R}') \\
&= a[\mathbf{R}', \mathbf{R}] \\
&= ja(1/\alpha)(\mathbf{A} - p\mathbf{R} - q\mathbf{I})
\end{aligned} \tag{33}$$

and

$$[\mathbf{R}', \mathbf{R}] = j(1/\alpha)(\mathbf{A} - p\mathbf{R} - q\mathbf{I}) \tag{34}$$

and

$$\begin{aligned}
(p\mathbf{R} + q\mathbf{I})^2 &= p^2(a\mathbf{R} + b\mathbf{I}) + 2pq\mathbf{R} + q^2\mathbf{I} \\
&= (ap^2 + 2pq)\mathbf{R} + (bp^2 + q^2)\mathbf{I} \quad .
\end{aligned} \tag{35}$$

Since $\mathbf{R}'$ is also 2×2, it must obey its characteristic equation, so that there must exist functions f and g, of z such that

$$\mathbf{R}'^2 = f\mathbf{I} + g\mathbf{R}' \quad . \tag{36}$$

However, if we premultiply Eq. (28) by $\mathbf{R}'$ and subtract it from Eq. (28) post-multiplied by $\mathbf{R}'$, we find that

$$\mathbf{R}\mathbf{R}'^2 = \mathbf{R}'^2\mathbf{R} \tag{37}$$

so that $\mathbf{R}'^2$ must commute with $\mathbf{R}$. If $\mathbf{R}'$ commutes with $\mathbf{R}$, the problem is trivial. It is simple to show, then, that either $\mathbf{R}$ is constant or $(a^2 + 4b)$ is zero. If $\mathbf{R}'$ does not commute with $\mathbf{R}$, then, substituting Eq. (36) in Eq. (37)

$$g\mathbf{R}\mathbf{R}' = g\mathbf{R}'\mathbf{R} \tag{38}$$

so that g must vanish, and

$$\mathbf{R}'^2 = f\mathbf{I} \quad . \tag{39}$$

Hence, Eq. (32) becomes

$$[\mathbf{R}', \mathbf{R}]^2 = (f/\alpha)\mathbf{I} \quad . \tag{40}$$

If we collect these results together by putting Eqs. (40), (33), (34), and (35) into Eq. (31), we find that the terms in $\mathbf{R}$ cancel and we are left with

$$\mathbf{A}^2 = (ap + 2q)\mathbf{A} + (bp_2 - q^2 - apq - \alpha f)\mathbf{I} \quad . \tag{41}$$

We can now choose p and q so that the coefficients are constants. One convenient class of solutions is obtained by setting

$$q = -\frac{1}{2}ap \quad . \tag{42}$$

Then

$$\mathbf{A}^2 + c\mathbf{I} = \mathbf{0} \tag{43}$$

where

$$c = -bp^2 + q^2 + apq + \alpha f$$

$$= -\frac{1}{4}(a^2 + 4b)p^2 + \alpha f$$

$$= + (p^2/4\alpha) + \alpha f \tag{44}$$

and

$$p = \{4\alpha(c - \alpha f)\}^{1/2} \quad . \tag{45}$$

For an arbitrary constant c, then, Eqs. (45) and (42) determine a p and q such that **A**, from Eq. (26), has constant eigenvalues and is expandable in the form of Eq. (7).

3. R Isomorphic to the Rotation Group

We shall now consider the problem in a still more abstract manner so as to obtain greater generality. Specifically, we shall find that we can obtain these same results if **R** is embedded in a representation of the rotation group.

We shall not here discuss in detail what is the significance of requiring that **R** be embedded in a representation of the rotation group. For this, reference may be made to Interim Report 3, Sec. I. We shall only assert the requirement that there shall exist the set of infinitesimal transformations, $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ such that **R** is expandable in terms of them:

$$\mathbf{R} = \gamma_1\mathbf{M}_1 + \gamma_2\mathbf{M}_2 + \gamma_3\mathbf{M}_3 \tag{46}$$

where $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ are constant so that the entire z-dependence of **R** is contained in the scalars γ_1, γ_2, and γ_3. We require also that these infinitesimal transformation be such that they obey the commutation rules,

$$[\mathbf{M}_1\ \mathbf{M}_3] = -\mathbf{M}_1$$
$$[\mathbf{M}_2\ \mathbf{M}_3] = \mathbf{M}_2$$
$$[\mathbf{M}_1\ \mathbf{M}_2] = -2\mathbf{M}_3 \quad . \tag{47}$$

We can also add to Eq. (46) any constant matrix without changing what follows to any important degree, although it does complicate the procedures necessary to obtain an **A** with constant eigenvalues. (See the Appendix of Interim Report 3 for the elimination of such terms.)

It should be noted that the requirement that **R** be embedded in a representation of the rotation group includes cases in which **R** is of any desired dimensionality. As discussed in Interim Report 3, Sec. I, there are representations of the rotation group involving $n \times n$ matrices, where n is any positive integer. The requirement does imply, as indicated by Eq. (46), that the z-dependence of **R** shall not require more than three independent functions of z, but this is all.

Since the **M**'s of Eq. (46) are constant, we have from Eq. (46) that

$$\frac{d\mathbf{R}}{dz} = \delta_1 \mathbf{M}_1 + \delta_2 \mathbf{M}_2 + \delta_3 \mathbf{M}_3 \tag{48}$$

where

$$\delta_i = d\gamma_i/dz \quad . \tag{49}$$

Our natural course of action at this point would be to determine how we may assure that **R** has constant eigenvalues. It is not evident how to do this, however, and we will have to obtain this condition indirectly.

We assume, now that **A** is also expressible in the **M**'s,

$$\mathbf{A} = \alpha_1 \mathbf{M}_1 + \alpha_2 \mathbf{M}_2 + \alpha_3 \mathbf{M}_3 \quad . \tag{50}$$

TThen we find that

$$\begin{aligned}
[\mathbf{A}, \mathbf{R}] &= (\alpha_1\gamma_2 - \alpha_2\gamma_1)[\mathbf{M}_1, \mathbf{M}_2] + (\alpha_1\gamma_3 - \alpha_3\gamma_1)[\mathbf{M}_1, \mathbf{M}_3] \\
&\quad + (\alpha_2\gamma_3 - \alpha_3\gamma_2)[\mathbf{M}_2, \mathbf{M}_3] \\
&= -2(\alpha_1\gamma_2 - \alpha_2\gamma_1)\mathbf{M}_3 - (\alpha_1\gamma_3 - \alpha_3\gamma_1)\mathbf{M}_1 \\
&\quad + (\alpha_2\gamma_3 - \alpha_3\gamma_2)\mathbf{M}_2
\end{aligned} \tag{51}$$

or that

$$\delta_1 = -j(-\alpha_1\gamma_3 + \alpha_3\gamma_1)$$

$$\delta_2 = -j(\alpha_2\gamma_3 - \alpha_3\gamma_2)$$

$$\delta_3 = -j(-2\alpha_1\gamma_2 + 2\alpha_2\gamma_1) \quad . \tag{52}$$

We find from Eq. (52), that

$$\delta_1\gamma_2 + \delta_2\gamma_1 - \frac{1}{2}\delta_3\gamma_3 = 0 \quad . \tag{53}$$

Equation (53) is the necessary and sufficient condition on $\mathbf{R}$ for Eq. (52) to have a non-trivial solution for the α's. It is therefore the necessary and sufficient condition that $d\mathbf{R}/dz$ be expressible as a commutator with an $\mathbf{A}$ expressible as Eq. (50), and this, therefore, is a sufficient condition for $\mathbf{R}$ to have constant eigenvalues.

Substituting Eq. (49) and integrating, we find that Eq. (53) requires that

$$\gamma_3^2 - 4\gamma_1\gamma_2 = \kappa \tag{54}$$

where η is constant. Equation (54) is also sufficient to assure that the eigenvalues of $\mathbf{R}$ are constant.

We have shown that if Eq. (54) is satisfied, a solution of Eq. (52) for the α's exists; the solution is not unique, however. One solution, as can be easily verified by substitution, is given by:

$$\alpha_1^0 = -jk(-\delta_1\gamma_3 - \delta_3\gamma_1) \tag{55}$$

$$\alpha_2^0 = -jk(\delta_2\gamma_3 - \delta_3\gamma_2) \tag{56}$$

$$\alpha_3^0 = -jk(-2\delta_1\gamma_2 + 2\delta_2\gamma_1) \tag{57}$$

where

$$k = -\frac{1}{\kappa} \quad . \tag{58}$$

Then

$$\mathbf{A}_0 = \alpha_1^0\mathbf{M}_1 + \alpha_2^0\mathbf{M}_2 + \alpha_3^0\mathbf{M}_3 \tag{59}$$

is a particular solution, although not, in general, one with constant eigenvalues.

The matrix $\mathbf{A}$ will now be expandable as Eq. (7) if its coefficients satisfy the equivalent of Eq. (54):

$$\alpha_3^2 - 4\alpha_1\alpha_2 = \mu \tag{60}$$

where μ is constant.

For this purpose, we let

$$\begin{aligned}
\alpha_1 &= (j/\kappa)(-\delta_1\gamma_3 + \delta_3\gamma_1) + p\gamma_1 \\
\alpha_2 &= (j/\kappa)(\delta_2\gamma_3 - \delta_3\gamma_2) + p\gamma_2 \\
\alpha_3 &= (j/\kappa)(-2\delta_1\gamma_2 + 2\delta_2\gamma_1) + p\gamma_3 \ .
\end{aligned} \tag{61}$$

That is, we let $\mathbf{A}$ be $(\mathbf{A}_0 + p\mathbf{R})$ where p is a scalar function of z, to be determined. The addition of $p\mathbf{R}$ to $\mathbf{A}_0$ does not affect the commutator.

We note that

$$\begin{aligned}
&(-2\delta_1\gamma_2 + 2\delta_2\gamma_1)^2 - 4(-\delta_1\gamma_3 + \delta_3\gamma_1)(\delta_2\gamma_3 - \delta_3\gamma_2) \\
&= -(\delta_3^2 - 4\delta_1\delta_2)(\gamma_3^2 - 4\gamma_1\gamma_2) + 4\left(\delta_1\gamma_2 + \delta_2\gamma_1 - \frac{1}{2}\delta_3\gamma_3\right)^2 \\
&= -\kappa(\delta_3^2 - 4\delta_1\delta_2)
\end{aligned} \tag{62}$$

so that

$$2\gamma_3(-2\delta_1\gamma_2 + 2\delta_2\gamma_1) - 4\gamma_2(-\delta_1\gamma_3 + \delta_3\gamma_1) - 4\gamma_1(\delta_2\gamma_3 - \delta_3\gamma_2) = 0 \ . \tag{63}$$

Hence

$$\alpha_3^2 - 4\alpha_1\alpha_2 = (1/\kappa)(\delta_3^2 - 4\delta_1\delta_2) + \kappa p^2 \tag{64}$$

and μ is constant if we set

$$p = \{(\mu/\kappa) - (\delta_3^2 - 4\delta_1\delta_2)/\kappa^2\}^{1/2} \quad . \tag{65}$$

With this function, **A** has eigenvalues and its derivative is expressible in the form of Eq. (7). Since **A** is, now, expressed as an isomorphism of the rotation group, the process can be continued.

The only limiting condition, other than those already stated, is that κ shall not anywhere vanish.

4. R Expanded in Pauli Spin Matrices

It may also be useful, at times, to have these results expressed in terms of the Pauli spin matrices. These may be taken as the matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{66}$$

The particular form of Eq. (66) is not significant for our purposes here. What is significant is the commutation rules.

$$[\sigma_1, \sigma_2] = -2j\sigma_3$$

$$[\sigma_2, \sigma_3] = -2j\sigma_1$$

$$[\sigma_3, \sigma_1] = -2j\sigma_2$$

$$[\sigma_1, \sigma_4] = [\sigma_2, \sigma_4] = [\sigma_3, \sigma_4] = 0 \quad . \tag{67}$$

For a discussion of the Pauli spin operators, the reader should consult any text on quantum mechanics, such as Dicke and Wittke, *Introduction to Quantum Mechanics* (Addison-Wesley Publishing Company, Inc., 1960), Chapter 12. The definitions of these matrices sometimes vary in the signs that are used, so that the reader should take care to note the particular definition being used. The reader should also note that the σ_4 defined in Eq. (66) is not properly a spin matrix, but is the identity element. Its inclusion, however, makes the set complete, allowing the representation of an arbitrary 2 × 2 matrix in the form

$$\mathbf{R} = r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3 + r_4\sigma_4 \tag{68}$$

where the coefficients can be determined by the orthogonality relations

$$\text{trace}\ (\sigma_i\sigma_j) = 2\delta_{ij}$$

so that

$$r_i = \frac{1}{2}\ \text{trace}\ (\sigma_i\mathbf{R})$$

where the trace of a matrix is the sum of the diagonal elements. (This procedure is based on the inner product relation defined in Appendix B, Sec. 3).

It may also be observed that it is possible to define a similar set of operators for any representation of the rotation group. That is, we can define matrices σ_1, σ_2, σ_3, such that the commutation relations of Eq. (67) are obtained. These matrices then provide a basis in terms of which we can expand any $\mathbf{R}$ that is embedded in this representation. Hence the procedures used here are not limited to 2 × 2 matrices. The development that follows is applicable to any system for which the development of the preceeding section is applicable. The two analyses are completely equivalent, although there may be considerable difference of convenience.

We consider now, an $\mathbf{R}$ that is expressed in the form of Eq. (68). Since the σ are constant, the derivative of $\mathbf{R}$ is:

$$\mathbf{R}' = r_1'\sigma_1 + r_2'\sigma_2 + r_3'\sigma_3 + r_4'\sigma_4 \quad . \tag{69}$$

We seek an $\mathbf{A}$ of the form

$$\mathbf{A} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 + a_4\sigma_4 \tag{70}$$

such that

$$\begin{aligned}\mathbf{R}' = -j[\mathbf{A}, \mathbf{R}] &= -j(a_1r_2 - a_2r_1)[\sigma_1, \sigma_2] \\ &\quad -j(a_1r_3 - a_3r_1)[\sigma_1, \sigma_3] \\ &\quad -j(a_2r_3 - a_3r_2)[\sigma_2, \sigma_3] \\ &= -2(a_1r_2 - a_2r_1)\sigma_3 \\ &\quad + 2(a_1r_3 - a_3r_1)\sigma_2 \\ &\quad - 2(a_2r_3 - a_3r_2)\sigma_1 \quad .\end{aligned} \tag{71}$$

Hence, we must have

$$\begin{aligned} r_1' &= 2(a_3r_2 - a_2r_3) \\ r_2' &= 2(a_1r_3 - a_3r_1) \\ r_3' &= 2(a_2r_1 - a_1r_2) \end{aligned} \tag{72}$$

Furthermore, r_4' must vanish, so that r_4 can, at most, be constant. For this to have a non-trivial solution for a_1, a_2, a_3, we must have

$$r_1r_1' + r_2r_2' + r_3r_3' = 0 \tag{73}$$

or

$$r_1^2 + r_2^2 + r_3^3 = A^2 \tag{74}$$

where A is a constant, not necessarily real.

We try

$$
\begin{aligned}
a_1 &= 2k(r_2 r'_3 - r_3 r'_2) \\
a_2 &= 2k(r_3 r'_1 - r_1 r'_3) \\
a_3 &= 2k(r_1 r'_2 - r_2 r'_1)
\end{aligned}
\tag{75}
$$

obtained from Eq. (72) by interchanging a_i and r'_i and multiplying by k. Substituting in Eq. (72) and using Eq. (63), we find that this is a solution for $\mathbf{A}$ providing $k = -1/(4A^2)$. Hence the "general" solution is

$$
\begin{aligned}
\mathbf{A} = & \left\{ \frac{-1}{2A^2} (r_2 r'_3 - r_3 r'_2) + f r_1 \right\} \sigma_1 \\
& + \left\{ \frac{-1}{2A^2} (r_3 r'_1 - r_1 r'_3) + f r_2 \right\} \sigma_2 \\
& + \left\{ \frac{-1}{2A^2} (r_1 r'_2 - r_2 r'_1) + f r_3 \right\} \sigma_3 \quad .
\end{aligned}
\tag{76}
$$

For $\mathbf{A}'$ to be expandable as in Eq. (7)—*i.e.*, as a commutator—the coefficients of $\mathbf{A}$ must satisfy an equation similar to Eq. (74). Using Eqs. (73) and (74), we find that

$$
\begin{aligned}
& (r_2 r'_3 - r_3 r'_2)^2 + (r_3 r'_1 - r'_1 r'_3)^2 + (r_1 r'_2 - r_2 r'_1)^2 \\
= \ & A^2 (r'^2_1 + r'^2_2 + r'^2_3) - (r_1 r'_1 + r_2 r'_2 + r_3 r'_3)^2 \\
= \ & A^2 (r'^2_1 + r'^2_2 + r'^2_3) \quad .
\end{aligned}
\tag{77}
$$

Also

$$
r_1(r_2 r'_3 - r_3 r'_2) + r_2(r_2 r'_1 - r_1 r'_3) + r_3(r_1 r'_2 - r_2 r'_1) = 0 \quad . \tag{78}
$$

Hence the condition that A be expandable requires that

$$\frac{1}{4A^2}\left(r_1'^2 + r_2'^2 + r_3'^2\right) + A^2 f^2 = B^2 \tag{79}$$

so that f is determined.

It should be noticed that we have used only the commutation relations of Eq. (67). Hence, the results are not limited to the specific form of Eq. (66). Further, it is not even limited to the dimensionality of Eq. (66). Specifically, if we are given a set $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ that obey Eq. (47), then it will be seen that

$$\begin{aligned}
\sigma_1 &= \mathbf{M}_1 - \mathbf{M}_2 \\
\sigma_2 &= j(\mathbf{M}_1 + \mathbf{M}_2) \\
\sigma_3 &= 2\mathbf{M}_3 \\
\sigma_4 &= \mathbf{I}
\end{aligned} \tag{80}$$

have the commutation rules of Eq. (67). Hence, any system that is isomorphic to the rotation group can be handled in this formalism.

5. Conclusions

We have here developed the analytic technique for finding an $\mathbf{A}$ that is expandable as Eq. (7), given an $\mathbf{R}$ that is expandable as Eq. (6). We have done this for various types of systems which describe the effect of coupling modes, or systems of modes, together by pairs.

The three cases considered are: (1) R a general 2×2 matrix, (2) R expressible as a representation of the three-dimensional rotation group, and (3) R expressed in terms of either the Pauli or the generalized spin matrices. Cases (2) and (3) are equivalent problems, being merely different representations, and Case (1) is a subclass of the others. They have been given in detail since one or another of the solutions may be most convenient in different situations, and in the hope that general insight will be gained by studying these cases in detail.

C. AN EXPLICIT SOLUTION FOR A FOR R OF SIMPLE STRUCTURE

In this section we shall seek a solution of Eq. (2) for **A**, given **R** and **R'**, that will involve only **R** and **R'** and the eigenvalues of **R**. That is, we shall seek to avoid explicit use of the eigenvectors of **R**. The reasons for seeking such a solution have been outlined in the introduction.

We shall limit ourselves here to **R** of simple structure. That is, we shall assume **R** to have a complete set of eigenvectors, so that we do not need to use generalized eigenvectors. This does not appear to be a serious restriction since most practical systems do involve a system matrix that is of simple structure except under certain limiting conditions such as a band-pass filter on the edge of the passband, or a TWT at the critical degree of asynchronism where amplification is lost.

A more general expression which does not require **R** to be of simple structure is developed in Appendix B, based on the abstract analysis of Appendix A. Hence, even this restriction can be avoided.

We have the problem then of solving Eq. (2) for **A**. We note that Eq. (2) is of the form

$$\mathbf{V} = j[\mathbf{R}, \mathbf{U}] \quad . \tag{81}$$

We can consider $j[\mathbf{R}, \ ]$ as an operator that maps **U** onto **V**. We wish to find an operator that will map **V** onto **U**. That is, we wish to find the inverse operator.

The operator $j[\mathbf{R}, \ ]$, however, is singular. It "annihilates" any **U** that commutes with **R**—*e.g.*, **R** itself. Therefore, it has no inverse. It has no inverse since the range, S_1, of **V**, is a proper subspace,[2] and not the whole space, S, of $n \times n$ matrices.

What we need is not a true inverse, because **V** must be in S_1, and not S. What we need is an operator, whose domain[2] is S_1, and which

[2] The set of elements of the whole space for which an operator is defined is called the "domain" of the operator. The space spanned by the result of applying an operator to any element within its domain is the "range" of the operator.

acts as an inverse within this domain. What it does to a **V** outside this domain is of no significance since **V** is necessarily within S_1.[3]

The problem, then, is to determine a "restricted inverse" to $j[\mathbf{R},\ ]$, whose domain is the range of $j[\mathbf{R},\ ]$. This operator, then, will not determine all **U** that give a possible **V**, but instead will give one possible **U**. It is in this sense that it is an inverse operator.

We note that $j[\mathbf{R},\ ]$ is a linear operator; that is, if k_1 and k_2 are any two scalars and $\mathbf{U}_1$ and $\mathbf{U}_2$ are any two matrices in the range of $j[\mathbf{R},\ ]$,

$$j[\mathbf{R},\ (k_1\mathbf{U}_1 + k_2\mathbf{U}_2)] = k_1 j[\mathbf{R},\ \mathbf{U}_1] + k_2 j[\mathbf{R},\ \mathbf{U}_2] \tag{82}$$

Therefore, $j[\mathbf{R},\ ]$ has a minimum polynomial. That is, if **T** is the operator,

$$\mathbf{T} = j[\mathbf{R},\ ] \tag{83}$$

there exist polynomials in **T**, $\phi(\mathbf{T})$, such that

$$\phi(\mathbf{T})\mathbf{U} = 0 \tag{84}$$

for all **U**. The polynomial of lowest degree is, then, the "minimum polynomial."

To interpret Eq. (84), we must understand what is meant by a power of **T**. Clearly this involves the successive application of **T**, leading to the k-commutators. That is, we define

$$[\mathbf{R},\ \mathbf{U}] = \mathbf{RU} - \mathbf{UR}$$

$$[_2\mathbf{R},\ \mathbf{U}] = [\mathbf{R}, [\mathbf{R},\ \mathbf{U}]]$$

$$[_k\mathbf{R},\ \mathbf{U}] = [\mathbf{R}[_{k-1}\mathbf{R},\ \mathbf{U}]] \tag{85}$$

[3] This statement of the problem, and some of the argument that follows, imply that it is possible to consider the set of all $n \times n$ matrices as a linear vector space. This is a valid viewpoint, although somewhat unusual. It depends upon the fact that the set of $n \times n$ matrices obey the postulates of an abstract linear vector space, for which the reader should consult any text on abstract linear algebra, such as *Linear Algebra and Matrix Theory*, by R. R. Stoll, (McGraw-Hill Book Co., New York, N.Y., 1952) Sec. 2.2 *et. seq.* In Appendix B, we shall exhibit an isomorphism between certain classes of **R** and particular vector spaces that are expressed in the usual form as a column matrix.

and interpret $\mathbf{T}^k(\mathbf{U})$ as

$$\mathbf{T}^k(\mathbf{U}) = j^k[{}_k\mathbf{R}, \mathbf{U}] \quad . \tag{86}$$

In the sequence of k-commutators, it is customary to define

$$[{}_0\mathbf{R}, \mathbf{U}] = \mathbf{U}$$

so that $\mathbf{T}^0(\mathbf{U})$would equal $\mathbf{U}$.

However, Eq. (84) cannot then contain $\mathbf{T}^0$ since a $\mathbf{U}$ that is annihilated by all $\mathbf{T}^k$ with $k \geq 0$—*e.g.*, $\mathbf{R}$—is not annihilated by $\mathbf{T}^0$. Hence $\phi(\mu)$, the polynomial function of μ from which $\phi(\mathbf{T})$ is derived, cannot contain any constant term.

We will suppose, however, that $\phi(\mu)$ does contain a term in μ, and will "normalize" the polynomial so that the coefficient of this term is minus unity. The significance of this assumption will be investigated later. For the moment we will simply assert that it appears to be equivalent to the assertion that $\mathbf{R}$ is of simple structure.

We will also assert that $\phi(\mu)$ contains only odd powers of μ. This we will later show to be true.

It follows, then, that ϕ can be written as

$$\phi(\mu) = \{\mu\psi(\mu) - 1\}\mu \tag{87}$$

or that

$$\psi(\mathbf{T})\mathbf{T}^2\mathbf{X} = \mathbf{T}\mathbf{X} \tag{88}$$

for any matrix $\mathbf{X}$.

If this is true, then

$$\mathbf{U}_0 = \psi(\mathbf{T})\mathbf{V} \tag{89}$$

is a solution of Eq. (81), where we now write Eq. (81) in the form,

$$\mathbf{V} = \mathbf{T}\mathbf{U} \quad . \tag{90}$$

We can verify this by substituting Eq. (89) in Eq. (90) and using Eq. (88). Since $\mathbf{V}$ is not annihilated by $\mathbf{T}$, for $\mathbf{R}$ of simple structure, we can, for this operand, cancel a $\mathbf{T}$ out of Eq. (88), so that Eq. (88) reduces to

$$\psi(\mathbf{T})\mathbf{T}\mathbf{V} = \mathbf{V} \quad . \tag{91}$$

Equation (89) then gives us the particular $\mathbf{U}$, $\mathbf{U}_0$, for a given $\mathbf{V}$, that is within the range of $\mathbf{T}$. However, any $\mathbf{U}$ whose projection on the range is $\mathbf{U}_0$, is also a solution. We may write, as the general solution,

$$\mathbf{U} = \mathbf{U}_0 + \xi(z, \mathbf{R}) \tag{92}$$

where $\xi(z, x)$ is any suitable function of x and z. [It is suitable, again, if, for any value of z, $\xi(z, x)$ is analytic in x in a simply connected region that includes all the eigenvalues of $\mathbf{R}$.] Since the added factor is annihilated by $\mathbf{T}$, the solution remains valid.

[The particular form of Eq. (92) may not be the most general possible solution if $\mathbf{R}$ is degenerate. It appears to be sufficiently general for our purposes, however.]

1. The Determination of ϕ

We must now investigate the significance of the polynomial $\phi(\mu)$.

We will suppose, throughout this section that $\mathbf{R}$ is of simple structure. We do not require that it be non-degenerate, so that some of its eigenvalues may be multiple, but we do require that there be a complete set of linearly independent eigenvectors. Suppose these are the set $\mathbf{x}_i$ such that

$$\mathbf{R}\mathbf{x}_i = \lambda_i \mathbf{x}_i \quad . \tag{93}$$

The operation of any matrix $\mathbf{U}$ on $\mathbf{x}_i$ yields a vector that can, in turn, be expanded in terms of the set

$$\mathbf{U}\mathbf{x}_i = \sum_j u_{ij}\mathbf{x}_j \quad . \tag{94}$$

If we consider the operation of the commutator on $\mathbf{x}_i$, we find

$$[\mathbf{R}, \mathbf{U}]\mathbf{x}_i = (\mathbf{RU} - \mathbf{UR})\mathbf{x}_i$$

$$= \mathbf{R} \sum u_{ij}\mathbf{x}_j - \lambda_i \mathbf{U}\mathbf{x}_i$$

$$= \sum(\lambda_j u_{ij}\mathbf{x}_j - \lambda_i u_{ij}\mathbf{x}_j)$$

$$= \sum_j(\lambda_j - \lambda_i)u_{ij}\mathbf{x}_j \quad .$$

We may guess that the effect of the p-commutator on $\mathbf{x}_i$ is:

$$[{}_p\mathbf{R}, \mathbf{U}]\mathbf{x}_i = \sum_j(\lambda_j - \lambda_i)^p u_{ij}\mathbf{x}_j \ . \tag{95}$$

This we can prove by induction. It is true for $p = 1$. Assume it is true for p. Then we find that

$$[{}_{p+1}\mathbf{R}, \mathbf{U}]\mathbf{x}_i = \mathbf{R}[{}_p\mathbf{R}, \mathbf{U}]\mathbf{x}_i - [{}_p\mathbf{R}, \mathbf{U}]\mathbf{R}\mathbf{x}_i$$

$$= \mathbf{R}\sum(\lambda_j - \lambda_i)^p u_{ij}\mathbf{x} - \lambda_i\sum(\lambda_j - \lambda_i)^p u_{ij}\mathbf{x}_j$$

$$= \sum(\lambda_j - \lambda_i)^{p+1}u_{ij}\mathbf{x}_j \quad .$$

Hence Eq. (95) is proven. Therefore, we find that, for a polynomial, $\phi(\mu)$ in μ:

$$\phi(\mu) = a_0 + a_1\mu + a_2\mu^2 + \ldots a_k\mu^k \tag{96}$$

$$\phi(\mathbf{T})\mathbf{U}\mathbf{x}_i = \sum j^p a_p [{}_p\mathbf{R}, \mathbf{U}]\mathbf{x}_i$$

$$= \sum_h\{\sum_p j^p a_p(\lambda_h - \lambda_i)^p\}u_{ih}\mathbf{x}_h \quad . \tag{97}$$

If, now, we set

$$\mu_{ih} = j(\lambda_h - \lambda_i) \tag{98}$$

then

$$\phi(\mathbf{T})\mathbf{U}\mathbf{x}_i = \Sigma\phi(\mu_{ih})u_{ih}\mathbf{x}_h . \tag{99}$$

This will vanish for all $\mathbf{x}_i$ providing $\phi(\mu)$ is a polynomial whose roots are the set μ_{ih}. It will, furthermore, be the minimum polynomial if its roots are simple, so that it has the form

$$\phi(\mu) = \prod_{ij}(\mu - \mu_{ij}) \tag{100}$$

where i, j take only such values that the set μ_{ij} is distinct.

We note that $\mu_{ii} = 0$ so that the polynomial so determined has no constant term, as required. We also note that $\mu_{ji} = -\mu_{ij}$, so that for each non-zero root there is another that is its negative. Hence, the non-zero terms generate a polynomial in even powers of μ. Multiplying by μ for the factor corresponding to the zero root, we find that $\phi(\mu)$ contains only odd powers of μ, as required.

The coefficient of the first power of μ is the product of all the non-zero μ_{ij}. Hence, the minimum polynomial normalized according to Eq. (87) is

$$\phi(\mu) = \frac{-\prod_{ij}(\mu - \mu_{ij})}{\prod'_{ij}\mu_{ij}} \tag{101}$$

where i, j take values such that each possible μ_{ij} occurs just once, and where the prime on the product symbol in the denominator indicates that the factor $\mu_{ij} = 0$ is omitted from the product.

We note that we cannot have all eigenvalues of $\mathbf{R}$ the same. Since $\mathbf{R}$ is assumed to be of simple structure, it would then be a scalar function times the identity. But the constancy of the eigenvalues would require that the scalar function be constant. Hence, the whole problem becomes trivial. Excluding this trivial case, there must exist a set of non-zero μ_{ij}.

We conclude, then, that a $\phi(\mu)$ of the form specified in Eq. (101) must exist for any $\mathbf{R}$ of simple structure.

As a note of parenthetical interest, we can observe that $\phi(\mu)$ is the minimum polynomial of the matrix

$$(j\mathbf{I} \times \mathbf{R} - j\mathbf{R} \times \mathbf{I}) \tag{102}$$

where the symbol x indicates the "Kronecker" or "direct" product.[4]

D. THE CANONICAL TRANSFORMATIONS OF A CHAIN OF EQUATIONS

In this section we will consider the solution of the chain of equations:

$$\begin{aligned}
\frac{d\mathbf{A}}{dz} &= 0 & \mathbf{A} &= \mathbf{A}_0 \\
\frac{d\mathbf{B}}{dz} &= j[\mathbf{B}, \mathbf{A}] & \mathbf{B}(0) &= \mathbf{B}_0 \\
\frac{d\mathbf{C}}{dz} &= j[\mathbf{C}, \mathbf{B}] & \mathbf{C}(0) &= \mathbf{C}_0
\end{aligned} \tag{103}$$

etc.

The class of non-uniform systems described by such a chain is that considered in Sec. I, and in Interim Report 2, Sec. II-A, and Interim Report 3, Sec. II. This class is of great interest since it is a class that exhibits some of the potentialities of non-uniformity, but is at the same time analyzable. It includes, for example, the important class of parametrically coupled systems.

We will consider here the solution of such a chain. Specific solutions for the simpler cases have been given in Interim Report 3, Sec. II-D. We shall generalize the procedure used there and obtain a process that can be repeatedly applied to the chain to effect a reduction of its complexity.

It should be noted that, in Eq. (103), we have reversed the order of the chain from our usual designation. The matrix **A** is the one that

[4]See, for example R. Bellman, *Introduction to Matrix Analysis*, (McGraw-Hill Book Co., Inc. New York, N.Y., 1960) Chapter 12, or other standard text.

is constant. (We are not concerned here with chains that do not terminate.) The matrix **B** is the one which can be regarded as representing a simple rotation around a constant axis. The matrix **C** then represents a rotation around an axis that is itself rotating about a constant axis. And so on. (This geometric interpretation is valid, but it should be remembered that the vector space involved may have many dimensions and a not-positive-definite metric. In relativistic terms, some of the dimensions may be "time-like." Hence, the geometry of the space may be quite different from Euclidean space.) The system operator, in our usual sense of this term, is then the highest member of the chain in which we are interested.

This change of order is convenient for our purposes because our procedure will involve the successive isolation and elimination of the complexities introduced by the lower members of the chain. We shall, in the first step, change the basis to a rotating one, such that **B** becomes constant and **C** becomes a description of a simple rotation about a fixed axis. Then we shall make a second change of basis such that **C** becomes constant. And so on, up the chain. We shall finally obtain a set of equations for the changes of basis that will effect these simplifications. While these equations, for the higher terms, are not simple, they are in principle soluble. Thus, we shall establish a canonical procedure for the analysis of such a chain.

The only restriction that we shall impose, other than the implied one that the system matrix is expandable as a chain of finite length, is that system everywhere exhibits a known conservation law. As discussed, for example, in Technical Note 1, Appendix A, and Interim Report 3, Sec. II-B, this requires that the system matrix, **R**, be K-hermitian ($\mathbf{KR} = \mathbf{R}^{\dagger}\mathbf{K}$). And, as shown in Interim Report 3, Sec. II-C, Theorem 3, we can then choose all the members of the chain so that they are everywhere K-hermitian. We are then justified in assuming that all the matrices of Eq. (103) are everywhere K-hermitian.

1. First Step

We have specified the chain so that **A** is a constant matrix, $\mathbf{A}_0$. We can use this fact to obtain a z-dependent change of basis that will make **B** a constant.

We consider the change of basis induced by a transformation $\mathbf{U}_1$ that is K-unitary:

$$\mathbf{U}_1^\dagger \mathbf{K}\mathbf{U}_1 = \mathbf{K} \quad . \tag{104}$$

Evidently $\mathbf{U}_1$ is non-singular, since $\mathbf{K}$ is, and so may be used as the transformation matrix of a change of basis:

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{U}_1^{-1}\mathbf{A}\mathbf{U}_1 \\ \mathbf{B}_1 &= \mathbf{U}_1^{-1}\mathbf{B}\mathbf{U}_1 \\ &\text{etc.} \end{aligned} \tag{105}$$

If, now, we define

$$\mathbf{M}_1 = j\mathbf{U}_1^{-1}\frac{d\mathbf{U}_1}{dz} \tag{106}$$

then $\mathbf{M}_1$ is K-hermitian, since

$$\begin{aligned} \mathbf{K}^{-1}\mathbf{M}_1^\dagger\mathbf{K} &= -j\mathbf{K}^{-1}\frac{d\mathbf{U}_1^\dagger}{dz}\mathbf{U}_1^{\dagger\,-1}\mathbf{K} \\ &= -j\left(\frac{d\mathbf{U}_1^{-1}}{dz}\mathbf{K}^{-1}\right)\mathbf{K}\mathbf{U}_1 \\ &= +j\mathbf{U}_1^{-1}\frac{d\mathbf{U}_1}{dz}\mathbf{U}_1^{-1}\mathbf{U}_1 \\ &= +j\mathbf{U}_1^{-1}\frac{d\mathbf{U}_1}{dz} = \mathbf{M}_1 \quad . \end{aligned} \tag{107}$$

Furthermore, $\mathbf{B}_1$, $\mathbf{C}_1$, etc., are K-hermitian, since $\mathbf{B}$ is K-hermitian, and

$$\begin{aligned} \mathbf{K}^{-1}\mathbf{B}^\dagger\mathbf{K} &= \mathbf{K}^{-1}\mathbf{U}_1^\dagger\mathbf{B}^\dagger\mathbf{U}_1^{\dagger\,-1}\mathbf{K} \\ &= \mathbf{U}_1^{-1}\mathbf{K}^{-1}\mathbf{B}^\dagger\mathbf{K}\mathbf{U}_1 = \mathbf{U}_1^{-1}\mathbf{B}\mathbf{U}_1 = \mathbf{B}_1 \end{aligned} \tag{108}$$

Now differentiating the expression for $\mathbf{B}_1$ in Eq. (105), we find

$$\frac{d\mathbf{B}_1}{dz} = -\mathbf{U}_1^{-1}\frac{d\mathbf{U}_1}{dz}\mathbf{U}_1^{-1}\mathbf{B}\mathbf{U}_1 + \mathbf{U}_1^{-1}\mathbf{B}\frac{d\mathbf{U}_1}{dz} + \mathbf{U}_1^{-1}\frac{d\mathbf{B}}{dz}\mathbf{U}_1$$

$$= j\mathbf{M}_1\mathbf{B}_1 - j\mathbf{B}_1\mathbf{M}_1 + j[\mathbf{B}_1, \mathbf{A}_1]$$

$$= j[\mathbf{B}_1, (\mathbf{A}_1 - \mathbf{M}_1)] \quad . \tag{109}$$

Similarly, we have that

$$\frac{d\mathbf{C}_1}{dz} = j[\mathbf{C}_1, (\mathbf{B}_1 - \mathbf{M}_1)]$$

$$\frac{d\mathbf{D}_1}{dz} = j[\mathbf{D}_1, (\mathbf{C}_1 - \mathbf{M}_1)]$$

etc. (110)

That is, if the original sequence was K-hermitian, and if we choose $\mathbf{U}_1$ to be K-unitary, then the new sequence, $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$, etc., are still K-hermitian.

We were given that $\mathbf{A}$ was constant. If, then, we set

$$\mathbf{M}_1 = \mathbf{A}_1 = \mathbf{A}_0\mathbf{U}_1(0) = \mathbf{I} \tag{111}$$

then Eq. (109) reduces to $d\mathbf{B}_1/dz = \mathbf{0}$, and $\mathbf{B}_1$ is constant. To accomplish this, we require that

$$j\mathbf{U}_1^{-1}\frac{d\mathbf{U}_1}{dz} = \mathbf{U}_1^{-1}\mathbf{A}\mathbf{U}_1 = \mathbf{U}_1^{-1}\mathbf{A}_0\mathbf{U}_1 \tag{112}$$

or

$$\frac{d\mathbf{U}_1}{dz} = -j\mathbf{A}_0\mathbf{U}_1 \tag{113}$$

$$\mathbf{U}_1 = \exp(-j\mathbf{A}_0 z) \quad . \tag{114}$$

Furthermore, this $\mathbf{U}_1$ is indeed *K*-unitary since $\mathbf{A}_0$ is *K*-hermitian. It is also non-singular for any $\mathbf{A}_0$.

We have, then, replaced the original *K*-hermitian sequence by a new *K*-hermitian sequence in which $\mathbf{B}_1$ is now constant. Furtheremore, since the transformation is the identity one at $z = 0$, we must have

$$\mathbf{B}_1 = \mathbf{B}_0 \quad . \tag{115}$$

We can now continue the process.

2. Second Step

We consider, as before a *K*-unitary transformation matrix $\mathbf{V}_2$ and define the *K*-hermitian matrix $\mathbf{N}_2$ as

$$\mathbf{N}_2 = \mathbf{J}\mathbf{V}_2^{-1} \frac{d\mathbf{V}_2}{dz} \quad . \tag{116}$$

We now let

$$\begin{aligned} \mathbf{B}_2 &= \mathbf{V}_2^{-1}\mathbf{B}_1\mathbf{V}_2 \\ \mathbf{C}_2 &= \mathbf{V}_2^{-1}\mathbf{C}_1\mathbf{V}_2 \\ &\text{etc.} \end{aligned} \tag{117}$$

and

$$\mathbf{M}_2 = \mathbf{V}_2^{-1}\mathbf{M}_1\mathbf{V}_2 \quad . \tag{118}$$

Then

$$\begin{aligned} \frac{d\mathbf{C}_2}{dz} &= -\mathbf{V}_2^{-1} \frac{d\mathbf{V}_2}{dz} \mathbf{V}_2^{-1}\mathbf{C}_1\mathbf{V}_2 + \mathbf{V}_2^{-1}\mathbf{C}_1 \frac{d\mathbf{V}_2}{dz} + \mathbf{V}_2^{-1} \frac{d\mathbf{C}_1}{dz} \mathbf{V}_2 \\ &= j\mathbf{N}_2\mathbf{C}_2 - j\mathbf{C}_2\mathbf{N}_2 + j\mathbf{V}_2^{-1}[\mathbf{C}_1, (\mathbf{B}_1 - \mathbf{M}_1)]\mathbf{V}_2 \\ &= j[\mathbf{C}_2, (\mathbf{B}_2 - \mathbf{M}_2 - \mathbf{N}_2)] \end{aligned} \tag{119}$$

$$\frac{d\mathbf{D}_2}{dz} = j[\mathbf{D}_2, (\mathbf{C}_2 - \mathbf{M}_2 - \mathbf{N}_2)] \quad \text{etc.} \tag{120}$$

We choose $\mathbf{V}_2$ so that

$$\begin{aligned} \mathbf{N}_2 &= \mathbf{B}_2 - \mathbf{M}_2 \\ &= \mathbf{B}_2 - \mathbf{A}_2 \end{aligned} \qquad \mathbf{V}_2(0) = \mathbf{I} \tag{121}$$

or

$$j\mathbf{V}_2^{-1}\frac{d\mathbf{V}_2}{dz} = \mathbf{V}_2^{-1}(\mathbf{B}_1 - \mathbf{M}_1)\mathbf{V}_2 = \mathbf{V}_2^{-1}(\mathbf{B}_1 - \mathbf{A}_1)\mathbf{V}_2$$

$$\frac{d\mathbf{V}_2}{dz} = -j(\mathbf{B}_1 - \mathbf{M}_1)\mathbf{V}_2 = -j(\mathbf{B}_0 - \mathbf{A}_0)\mathbf{V}_0 \tag{122}$$

$$\mathbf{V}_2 = \exp\{-j(\mathbf{B}_0 - \mathbf{A}_0)z\} \quad . \tag{123}$$

This choice makes $\mathbf{C}_2$ a constant. At $z = 0$, both $\mathbf{V}_2$ and $\mathbf{U}_1$ are the identity, so the constant must be $\mathbf{C}_0$

$$\mathbf{C}_2 = \mathbf{C}_0 \quad . \tag{124}$$

3. Third Step

Continuing, we set

$$\mathbf{P}_3 = j\mathbf{W}_3^{-1}\frac{d\mathbf{W}_3}{dz} \tag{125}$$

$$\mathbf{C}_3 = \mathbf{W}_3^{-1}\mathbf{C}_2\mathbf{W}_3 \tag{126}$$

and $\mathbf{D}_3$ will be a constant providing we set

$$\mathbf{P}_3 = \mathbf{C}_3 - \mathbf{M}_3 - \mathbf{N}_3 \tag{127}$$

or

$$\frac{d\mathbf{W}_3}{dz} = -j(\mathbf{C}_2 - \mathbf{M}_2 - \mathbf{N}_2)\mathbf{W}_3$$

$$= -j(\mathbf{C}_2 - \mathbf{B}_2)\mathbf{W}_3$$

$$= -j(\mathbf{C}_0 - \mathbf{V}_2^{-1}\mathbf{B}_0\mathbf{V}_2)\mathbf{W}_3 \quad . \tag{128}$$

4. The Transformation Equations

The key equations can, then, be summarized as follows:

$$\frac{d\mathbf{U}_1}{dz} = -j\mathbf{A}_0\mathbf{U}_1 \qquad \mathbf{U}_1(0) = \mathbf{I} \tag{129}$$

$$\frac{d\mathbf{V}_2}{dz} = -j\{\mathbf{B}_0 - \mathbf{U}_1^{-1}\mathbf{A}_0\mathbf{U}_1\}\mathbf{V}_2 \qquad \mathbf{V}_2(0) = \mathbf{I} \tag{130}$$

$$\frac{d\mathbf{W}_3}{dz} = -j\{\mathbf{C}_0 - \mathbf{V}_2^{-1}\mathbf{B}_0\mathbf{V}_2\}\mathbf{W}_3 \qquad \mathbf{W}_3(0) = \mathbf{I} \tag{131}$$

$$\frac{d\mathbf{X}_4}{dz} = -j\{\mathbf{D}_0 - \mathbf{W}_3^{-1}\mathbf{C}_0\mathbf{W}_3\}\mathbf{X}_4 \qquad \mathbf{X}_4(0) = \mathbf{I} \tag{132}$$

etc.

It happens that, because of the simplicity of the first equation, $\mathbf{U}_1$ commutes with $\mathbf{A}_0$ and the second equation is simple. This is not true in the succeeding equations, however. Their complexity grows very rapidly.

If we write the general equation of this sequence of Eqs. (129), (130), (131), (132), etc. as

$$\frac{d\mathbf{R}}{dz} = -j\{\mathbf{F}_0 - \mathbf{S}^{-1}\mathbf{G}_0\mathbf{S}\}\mathbf{R} \tag{133}$$

we can make the substitution

$$\mathbf{R} = e^{-jF_0 z}\,\mathbf{R} \tag{134}$$

since $\mathbf{F}_0$ is constant, and obtain

$$\frac{d\mathbf{R}}{dz} = j\mathbf{S}^{-1}\mathbf{G}_0\mathbf{S}\mathbf{R} \tag{135}$$

where

$$\mathbf{S} = \mathbf{S}e^{-jF_0 z} \quad . \tag{136}$$

Hence, the entire system can be reduced to the form of Eq. (135). It is these equations for the appropriate basis-transformations that we consider the canonical equations for the original chain. The solution of the original chain has been made to depend on the solution to the canonical chain of equations of the form of Eq. (135).

III PROGRAM FOR THE NEXT INTERVAL

During the next interval work will continue on the problems studied here. It is thought that we are now in a position to begin the interpretation of some of these results in terms of physical processes that are of interest for devices.

One tool for this process is the "Baker-Hausdorff Formula." This is a formula giving the solution of

$$\exp C = (\exp A)(\exp B)$$

for C when A and B are non-commuting operators. Since the class of non-uniform systems that we have been considering are solved as the product of exponentials, the determination of the behavior of the system as a whole can be investigated conveniently with this formula.

We intend, also, to begin consideration of ways of developing a statistical theory of mode coupling. This is a new topic, but one that is important for many practical problems.

Finally, surveillance of the literature will be continued. No report on this aspect has been included in the present report, but will be continued in the next Interim Report.

APPENDIX A

THE "INVERSION" OF SINGULAR OPERATORS

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In this appendix we shall be concerned with the solution for $\mathbf{v}$ of the equation

$$\mathbf{u} = \mathbf{T}\mathbf{v} \tag{A-1}$$

when $\mathbf{u}$ is known. $\mathbf{T}$ is assumed to be a linear operator which may be singular.

A singular operator, of course, has no inverse. This is what we mean by calling it singular. Nevertheless, Eq. (A-1) is soluble in the sense that, for a given $\mathbf{u}$, $\mathbf{v}$ is determined within certain limits.

The singularity of $\mathbf{T}$ enters through the statement that not all $\mathbf{u}$ will generate an appropriate $\mathbf{v}$. However, this does not contradict the problem since the statement of the problem implies that $\mathbf{u}$ is in the range of $\mathbf{T}$. That is, if we can write Eq. (A-1) at all, then $\mathbf{u}$ must be in the subspace such that some $\mathbf{v}$ exists. The problem, therefore, is to find an operator that acts as an inverse for $\mathbf{T}$ when operating on the range of $\mathbf{T}$—*i.e.*, on the subspace that is obtained by operating on all vectors with $\mathbf{T}$. We do not care what this operator may do on any other vector.

The problem may arise in a number of different contexts. In vector analysis, we may meet the problem of finding $\vec{v}$, given $\vec{u}$, when

$$\vec{u} = \vec{a} \times \vec{v} \quad . \tag{A-2}$$

The operator $(\vec{a}\,\times)$ is easily seen to be a linear operator. It is a singular one since the cross product vanishes if $\vec{v}$ is a scalar times $\vec{a}$. And yet it may be important to be able to solve such an equation for $\vec{v}$.

Another problem that is of considerable interest is to find the matrix $\mathbf{V}$, given $\mathbf{U}$, when

$$\mathbf{V} = j\,[\mathbf{R},\mathbf{U}] = j\,(\mathbf{RU} - \mathbf{UR}) \tag{A-3}$$

i.e., when $\mathbf{V}$ is the commutator of $\mathbf{R}$ and $\mathbf{U}$. The space of $n \times n$ matrices can be considered a linear vector space. This may seem somewhat strange, but it can be easily seen that this space does obey the postulates of a linear vector space. The operator

$$\mathbf{T} = j[\mathbf{R},\ ] \qquad \text{(A-4)}$$

is a linear operator. It is, however, a singular one since $\mathbf{V}$ vanishes for any $\mathbf{U}$ that commutes with $\mathbf{R}$, such as $\mathbf{R}$ itself.

This form arises in various quantum mechanical problems. It also occurs in coupled mode theory. The class of systems with system matrix, $\mathbf{R}$, for which a matrix $\mathbf{A}$ exists such that

$$\frac{d\mathbf{R}}{dz} = j[\mathbf{R}, \mathbf{A}] \qquad \text{(A-5)}$$

is a class of "non-uniform" systems which are considerably simpler than the general non-uniform system. If $\mathbf{A}$ can be chosen to be constant, it is then possible to solve the system exactly and explicitly. This class, in fact, includes most of the solved non-uniform systems, such as the exponentially tapered transmission line and the distributed parametric amplifier. This is, then, a class of systems that justifies intensive study.

In the study of Eq. (A-5), our immediate concern is to determine if the system operator $\mathbf{R}(z)$ is governed by an equation of the form of Eq. (A-5). It can be shown that the necessary and sufficient condition for this is that the eigenvalues and structure of $\mathbf{R}$ be constant.

Having determined that this is true, our next concern is to determine what $\mathbf{A}$ may be, knowing $\mathbf{R}(z)$ and hence $d\mathbf{R}/dz$. This, then, is the problem of "inverting" the singular operator $j[\mathbf{R},\ ]$.

We note that we do not expect a unique solution to Eq. (A-5). Given an $\mathbf{A}$ that satisfies Eq. (A-5), we can, evidently, add to $\mathbf{A}$ any matrix function of z that everywhere commutes with $\mathbf{R}(z)$ without affecting Eq. (A-5). There is, then, a "pencil" of solutions.

In more abstract terms, given Eq. (A-1) with $\mathbf{T}$ a linear operator. If $\mathbf{v}$ is any solution to Eq. (A-1) for a given $\mathbf{u}$, and if $\mathbf{v}_0$ is any vector such that

$$\mathbf{T}\mathbf{v}_0 = 0 \tag{A-6}$$

then $(\mathbf{v} + \alpha\mathbf{v}_0)$ is a solution, α being any scalar.

Hence, if $\mathbf{T}$ is singular, the solution to Eq. (A-1) is not unique.

To discuss this problem we shall first discuss these complications in a precise way.

1. ABSTRACT FORMULATION

We will start by defining certain important concepts.

Definition: The "domain," S_0, of an operator, $\mathbf{T}$, is the subspace of vectors such that the operation of $\mathbf{T}$ is defined.

Normally, we consider the domain of a matrix operator to be the whole space. We do this because there is really no reason to do otherwise. (This is not true in infinitely dimensional spaces with either infinite matrices or integral or differential operators. In such spaces, it may not be possible to apply a given operator to any operand without introducing divergencies. It is then necessary to restrict the domain to assure convergence. In finitely dimensional spaces, such as is our concern, this problem does not arise so that we usually have no reason to specify the domain as other than the whole space.) However, in the present situation, we are seeking an operator whose operand is known to be confined to a given subspace. This operator will have significant meaning only for the domain that is spanned by the possible vectors $\mathbf{u}$.

Definition: The "range," S_R, of an operator, $\mathbf{T}$, is the subspace spanned by the vectors obtained by $\mathbf{T}$ operating on any vector in its domain. We can express this formally by writing

$$S_R = \mathbf{T}S_D$$

or by saying that the range of $\mathbf{T}$ is $\mathbf{T}$ operating on its domain.

The singularity of $\mathbf{T}$ is expressed by saying that its range is a proper subspace—*i.e.*, is not the whole space.

Definition: The "null space," S_N, of an operator, $\mathbf{T}$, is the subspace of its domain such that $\mathbf{T}$ operating on any vector in S_N vanishes:

$$\mathbf{T}\mathbf{x} = 0 \quad \text{if } \mathbf{x} \text{ is in } S_N \quad .$$

The singularity of $\mathbf{T}$ can also be expressed by stating that the null space of $\mathbf{T}$ is a proper subspace—*i.e.*, not the space containing only the null vector.

Definition: A set of linearly independent vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are called "progenitors of the range of $\mathbf{T}$" if the set $\mathbf{Tx}_1, \mathbf{Tx}_2, \ldots \mathbf{Tx}_k$ form a basis for the range of $\mathbf{T}$.

Note that the specification that $\mathbf{Tx}_1, \ldots \mathbf{Tx}_k$ form a basis implies that (1) they are linearly independent, and (2) none is the null vector. Hence a set of progenitors must contain exactly k vectors, if k is the dimensionality of the range.

Further, the vectors $\mathbf{x}_1 \ldots \mathbf{x}_k$ must be linearly independent. For, if there existed a non trivial set of scalars C_i such that

$$\sum C_i \mathbf{x}_i = \mathbf{0} \tag{A-7}$$

then, since $\mathbf{T}$ is linear, we would have that

$$\mathbf{T}\sum C_i \mathbf{x}_i = \sum C_i (\mathbf{Tx}_i) = \mathbf{0} \tag{A-8}$$

and the set $\mathbf{Tx}_i$ would not be linearly independent.

Hence, any set of progenitors spans a subspace which we will call a "progenitor space." This space is not uniquely defined, if $\mathbf{T}$ is singular. It has, however, the virtue that the operation of $\mathbf{T}$ on any progenitor space is non-singular. That is, to every vector in any given progenitor space there corresponds one and only one vector in the range of $\mathbf{T}$, and vice versa.

In solving Eq. (A-1), then, we know that $\mathbf{u}$ is in the range of $\mathbf{T}$. We wish to specify, for each $\mathbf{u}$, a unique $\mathbf{v}$ that is a particular solution to Eq. (A-1). That is, we wish to specify a particular progenitor space. The mapping of the progenitor space onto the range is, then one-to-one. If, in addition, the progenitor space can be taken as identical to the range, then this mapping can be expressed as a non-singular operator with this subspace as its domain. The problem is then directly soluble, giving a particular solution to Eq. (A-1). The general solution is then obtained by adding to the particular solution any vector in the null space of $\mathbf{T}$.

This is not, however, always possible to do. If **T** has a chain of of eigenvectors and generalized eigenvectors with zero eigenvalue, then the eigenvector that heads the chain is both in the null space of **T** and in the range. It is in the null space since it is annihilated by **T**. It is in the range since it is generated by the generalized eigenvector of rank two. Since it is in the null space, it is not by itself a progenitor. Therefore, the progenitor space cannot be identical with the range. While the mapping of a progenitor space onto the range is still one-to-one, it is not a mapping of a subspace onto itself. In this case, we will have to use a more involved procedure.

Before going into the general procedure, however, we will consider a simpler case in which this problem does not arise.

2. T OF SIMPLE STRUCTURE

We will consider first the case when **T** is of simple structure. This is not the most general situation that avoids the difficulty cited in the previous section. We could admit non-simple structure involving chains with non-zero eigenvalues. However, it does appear to include most situations of interest. The remaining situations in general seem always to include at least the threat that a zero-eigenvalued chain may exist, so that it is then better to use the more general procedure that we will discuss shortly.

We have, then, the following theorem

Theorem 1: If **T** is of simple structure, the set of eigenvectors with non-zero eigenvalues are both a basis for the range and a set of progenitors—*i.e.*, a basis for a progenitor space.

This follows directly by taking as a basis for the whole space the whole set of eigenvectors, which is complete since **T** is of simple structure.

This theorem can also be stated as:

Corollary: If **T** is of simple structure, the range of **T** is a progenitor space of **T**.

Thus this tells us that, in Eq. (A-1), we can restrict **v** to be within the range of **T** without, by so doing, restricting the range of **u**.

We can, in other words, be certain that, for any $\mathbf{u}$ that can occur in Eq. (A-1)—*i.e.*, is in the range of $\mathbf{T}$—there exists an $\mathbf{x}$ such that

$$\mathbf{u} = \mathbf{T}^2\mathbf{x} \quad . \tag{A-9}$$

Then, the particular $\mathbf{v}$, $\mathbf{v}_1$ that is within this progenitor space is given by

$$\mathbf{v}_1 = \mathbf{T}\mathbf{x} \quad . \tag{A-10}$$

The minimum polynomial of $\mathbf{T}$—*i.e.*, the polynomial function that annihilates any vector in the whole space—can be written as

$$\Phi(\mathbf{T}) = \Pi(\mathbf{T} - \lambda_i\mathbf{I}) \tag{A-11}$$

where the product is taken over all *distinct* values of λ_i. That is, if λ_i is a multiple root, the factor $(\mathbf{T} - \lambda_i\mathbf{I})$ appears only once. That this is the minimum polynomial follows from the fact that $\mathbf{T}$ is of simple structure.

Since $\mathbf{T}$ is assumed singular, zero is an eigenvalue. Hence Φ contains as a single factor, $\mathbf{T}$ itself, and can be written as

$$\Phi(\mathbf{T}) = a_1\mathbf{T} + a_2\mathbf{T}^2 + \ldots \tag{A-12}$$

where

$$a_1 = \Pi'(-\lambda_i) \tag{A-13}$$

the prime indicating that the product is over all distinct, *non-zero* values of λ_i. Hence, a_1 is not zero, and $\Phi(\mathbf{T})$ can be written as

$$\begin{aligned}\Phi(\mathbf{T}) &= \{a_1 + (a_2 + a_3\mathbf{T} + \ldots)\mathbf{T}\}\mathbf{T} \\ &= a_1\{1 - \psi(\mathbf{T})\mathbf{T}\}\mathbf{T}\end{aligned} \tag{A-14}$$

where

$$\psi(\mathbf{T}) = -\frac{1}{a_1}(a_2 + a_3\mathbf{T} + \ldots \ ) \quad . \tag{A-15}$$

The function $\Phi(\mathbf{T})$ is a function that annihilates any vector, $\mathbf{x}$. Hence, for any $\mathbf{x}$, we have that

$$\mathbf{Tx} = \psi(\mathbf{T})\mathbf{T}^2\mathbf{x} \quad . \tag{A-16}$$

If, now, we premultiply Eq. (A-9) by $\psi(\mathbf{T})$, we find

$$\psi(\mathbf{T})\mathbf{u} = \psi(\mathbf{T})\mathbf{T}^2\mathbf{x} = \mathbf{Tx} = \mathbf{v}_1 \tag{A-17}$$

so that the particular solution is

$$\mathbf{v}_1 = \psi(\mathbf{T})\mathbf{u} \tag{A-18}$$

and the general solution is

$$\mathbf{v} = \psi(\mathbf{T})\mathbf{u} + \mathbf{v}_0 \tag{A-19}$$

where $\mathbf{v}_0$ is any vector in the null space to $\mathbf{T}$—*i.e.*, such that

$$\mathbf{Tv}_0 = \mathbf{0} \quad . \tag{A-20}$$

We have, then, obtained a general solution of Eq. (A-1) if $\mathbf{T}$ is of simple structure.

This may seem like a devious way of going about it. In particular, we have spent a considerable amount of time justifying the substitution of $\mathbf{Tx}$ for $\mathbf{v}$, which was then immediately removed from Eq. (A-17).

This was, however, necessary. It was necessary because the minimum polynomial of a singular operator contains no constant term. Therefore, we cannot find a function $\psi(\mathbf{T})$ such that, in general,

$$\mathbf{x} = \psi(\mathbf{T})\mathbf{Tx} \quad . \tag{A-21}$$

The best we can do is Eq. (A-16), in which the right side contains the factor $\mathbf{T}^2$. The manipulation through the vector $\mathbf{x}$ defined such that $\mathbf{Tx} = \mathbf{v}_1$, is necessary to take account of this factor.

If $\mathbf{T}$ is non-singular, we can be certain that the minimum polynomial does contain a non-zero constant term, and a $\psi(\mathbf{T})$ does exist such that

Eq. (A-21) is true. In this case, the solution to Eq. (A-1) is obtained by premultiplying Eq. (A-1) by this $\psi(\mathbf{T})$. This is a valid, if somewhat unorthodox way of obtaining $\mathbf{T}^{-1}$ for a non-singular operator in terms of the positive powers of $\mathbf{T}$.

If, now, $\mathbf{T}$ is not only singular but has a chain of length k with zero eigenvalue, then the minimum polynomial contains the factor $\mathbf{T}^k$. Hence, instead of Eq. (A-16), we obtain

$$\mathbf{T}^k\mathbf{x} = \psi(\mathbf{T})\mathbf{T}^{k+1}\mathbf{x} \qquad \text{(A-22)}$$

which does not permit us to use the same procedure. We could use it if we could make the substitution

$$\mathbf{v} = \mathbf{T}^k\mathbf{x} \quad . \qquad \text{(A-23)}$$

This, however, excludes the possibility that $\mathbf{v}$, and hence $\mathbf{u}$, are in the entire subspace spanned by the chain of generalized eigenvectors with zero eigenvalue. This subspace does include vectors that are in the range of $\mathbf{T}$, as discussed before. Therefore, the $\mathbf{v}$ so found is not a solution for any $\mathbf{u}$, but for a specifically restricted $\mathbf{u}$. It is for this reason that we must seek a more general approach to this problem.

3. GENERAL $\mathbf{T}$

We can handle a general operator, $\mathbf{T}$, including those with chains of generalized eigenvectors with zero eigenvalue, by consideration of the adjoint operator.

Before doing so, we define the orthogonal complement of a subspace.

Definition: Given a (proper) inner product relation, the "orthogonal complement" of a subspace, S_1, is the subspace of all vectors that are orthogonal to every vector in S_1. That is, $\mathbf{x}$ is in the orthogonal complement of S_1 if

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

for *every* $\mathbf{y}$ in S_1.

The definition implies that the orthogonal complement of a subspace is a subspace. It is not difficult to see that this is so (providing the whole space has a finite number of dimensions).

We shall, then, make use of the following theorem.

Theorem 2: Given a (proper) inner product relation, the range of $\mathbf{T}$ is the orthogonal complement of the null space of $\mathbf{T}^\#$, where $\mathbf{T}^\#$ is the operator that is adjoint to $\mathbf{T}$ under the given inner product relation.

This theorem, then, gives us a connection between the range of $\mathbf{T}$ and a subspace of the whole space that is the complete domain of $\mathbf{T}$. We shall see later how to use this connection.

Before proving the theorem we need to prove the "projection theorem:"

Theorem 3: Given a subspace, S_1, a vector $\mathbf{x}$ not in S_1, and a (proper) inner product relation, then there exists a unique vector, $\mathbf{w}$, in S_1, called the "projection of $\mathbf{x}$ on S_1," such that

$$\langle \mathbf{x} - \mathbf{w}, \mathbf{y} \rangle = 0 \qquad \text{(A-24)}$$

for any $\mathbf{y}$ in S_1.

Consider any $\mathbf{y}$ in S_1. Then, since $\mathbf{x}$ is not in S_1, $(\mathbf{x} - \mathbf{y})$ is not the null vector. Hence

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle \neq 0 \quad . \qquad \text{(A-25)}$$

This inner product, considered as a function of $\mathbf{y}$, must have a greatest lower bound. That is, there must exist some $\mathbf{y}_0$ such that

$$\langle \mathbf{x} - \mathbf{y}_0, \mathbf{x} - \mathbf{y}_0 \rangle = a \qquad \text{(A-26)}$$

where a is a positive real number, and such that for any $\mathbf{y}$

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle \geq \langle \mathbf{x} - \mathbf{y}_0, \mathbf{x} - \mathbf{y}_0 \rangle = a \quad . \qquad \text{(A-27)}$$

We do not, at this point, assert that $\mathbf{y}_0$ is unique, but only that at least one such $\mathbf{y}_0$ exists.

We shall show, now, that $\mathbf{y}_0$ is the projection.

Any vector $\mathbf{y}$ in S_1 can be expressed as a linear combination of $\mathbf{y}_0$ and some other vector $\mathbf{z}$:

$$\mathbf{y} = \mathbf{y}_0 + \alpha\mathbf{z} \qquad \text{(A-28)}$$

where α is a scalar. Then

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle = \langle \mathbf{x} - \mathbf{y}_0 - \alpha\mathbf{z}, \mathbf{x} - \mathbf{y}_0 - \alpha\mathbf{z} \rangle$$

$$= \langle \mathbf{x} - \mathbf{y}_0, \mathbf{x} - \mathbf{y}_0 \rangle - \alpha^* \langle \mathbf{z}, \mathbf{x} - \mathbf{y}_0 \rangle - \alpha \langle \mathbf{x} - \mathbf{y}_0, \mathbf{z} \rangle$$

$$+ |\alpha|^2 \langle \mathbf{z}, \mathbf{z} \rangle \geq \langle \mathbf{x} - \mathbf{y}_0, \mathbf{x} - \mathbf{y}_0 \rangle \quad . \qquad \text{(A-29)}$$

This must be true for any non-null $\mathbf{z}$ and any α. In particular, it must be true for

$$\alpha = \frac{\langle \mathbf{z}, \mathbf{x} - \mathbf{y}_0 \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle} = \frac{\langle \mathbf{x} - \mathbf{y}_0, \mathbf{z} \rangle^*}{\langle \mathbf{z}, \mathbf{z} \rangle} \quad . \qquad \text{(A-30)}$$

Substituting this in, we find that

$$-\frac{|\langle \mathbf{x} - \mathbf{y}_0, \mathbf{z} \rangle|^2}{\langle \mathbf{z}, \mathbf{z} \rangle} \geq 0 \quad . \qquad \text{(A-31)}$$

The left side is negative semi-definite. It cannot be greater than zero, so it must equal zero. Since $\mathbf{z}$ is non-null and the inner product relation is proper—*i.e.*, positive definite—we must have

$$\langle \mathbf{x} - \mathbf{y}_0, \mathbf{z} \rangle = 0 \quad . \qquad \text{(A-32)}$$

Since this is true for any $\mathbf{z}$ in S_1, the $\mathbf{y}_0$ determined as a vector giving the greatest lower bound in Eq. (A-26), is a projection as defined by the theorem.

Suppose, now, $\mathbf{y}_0$ were not uniquely defined. Suppose there existed a $\mathbf{y}_1$ also giving the greatest lower bound in Eq. (A-26). Then it, too, would obey Eq. (A-32):

$$\langle \mathbf{x} - \mathbf{y}_1, \mathbf{z} \rangle = 0 \tag{A-33}$$

for any $\mathbf{z}$ in S_1.

Subtracting Eq. (A-32) from Eq. (A-33), we obtain

$$\langle \mathbf{y}_0 - \mathbf{y}_1 \;\; \mathbf{z} \rangle = 0 \quad . \tag{A-34}$$

But $\mathbf{y}_0$ and $\mathbf{y}_1$ are both in S_1. Therefore, so is their difference. Since $\mathbf{z}$ is any vector in S_1, we can take it, in particular, as $\mathbf{y}_0 - \mathbf{y}_1$:

$$\langle \mathbf{y} \;\; - \mathbf{y}_1, \mathbf{y} \;\; - \mathbf{y}_1 \rangle = 0 \quad . \tag{A-35}$$

Hence $\mathbf{y}_0 - \mathbf{y}_1$ is the null vector, and $\mathbf{y}_1$ is identical to $\mathbf{y}_0$.

Therefore, a projection exists and is unique.

It may be noted that the theorem concerns a vector not in S_1. This restriction can be immediately removed, however. If $\mathbf{x}$ is in S_1, then we define it as its own projection, and the result becomes trivial.

We use this theorem, now, to prove the following:

Theorem 4: Given a (proper) inner product relation, the equation

$$\mathbf{u} = \mathbf{T}\mathbf{v} \tag{A-36}$$

has a solution for a given $\mathbf{u}$ if and only if $\mathbf{u}$ is orthogonal to every $\mathbf{w}$ that is a solution of

$$\mathbf{T}^{\#}\mathbf{w} = \mathbf{0} \quad . \tag{A-37}$$

To prove the necessity of the condition, consider any vector $\mathbf{x}$ and any $\mathbf{w}$ that is a solution of $\mathbf{T}^{\#}\mathbf{w} = \mathbf{0}$:

$$0 = \langle \mathbf{T}^{\#}\mathbf{w}, \mathbf{x} \rangle = \langle \mathbf{w}, \mathbf{T}\mathbf{x} \rangle \quad . \tag{A-38}$$

This is true for any $\mathbf{x}$, and, in particular, for $\mathbf{x} = \mathbf{v}$:

$$\langle \mathbf{w}, \mathbf{T}\mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{u} \rangle = 0 \quad . \tag{A-39}$$

Therefore, $\mathbf{u}$ is orthogonal to any $\mathbf{w}$ that is a solution of the homogeneous adjoint equation.

To prove the sufficiency of the condition, suppose that $\mathbf{u}$ is orthogonal to all solutions, $\mathbf{w}$, of the homogeneous adjoint equation. We ask, then, if it then follows that $\mathbf{u}$ is in the range of $\mathbf{T}$.

Now the range of $\mathbf{T}$ is a subspace since, if

$$\mathbf{u}_1 = \mathbf{T}\mathbf{v}_1$$

$$\mathbf{u}_2 = \mathbf{T}\mathbf{v}_2 \tag{A-40}$$

then, since $\mathbf{T}$ is linear,

$$\alpha\mathbf{u}_1 + \beta\mathbf{u}_2 = \mathbf{T}(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) \quad . \tag{A-41}$$

That is, if $\mathbf{u}_1$ and $\mathbf{u}_2$ are in the range of $\mathbf{T}$, then so is any linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. Hence, the range is a subspace.

Therefore, the projection theorem applies. That is, if $\mathbf{x}$ is any vector, then there exists a $\mathbf{u}_0$ in the range of $\mathbf{T}$ such that

$$\begin{aligned} \langle \mathbf{u} - \mathbf{u}_0, \mathbf{T}\mathbf{x} \rangle &= 0 \\ &= \langle \mathbf{T}^{\#}(\mathbf{u} - \mathbf{u}_0), \mathbf{x} \rangle \quad . \end{aligned} \tag{A-42}$$

This is true for all $\mathbf{x}$, and in particular for $\mathbf{x} = \mathbf{T}^{\#}(\mathbf{u} - \mathbf{u}_0)$. Hence, this vector must be the null vector.

$$\mathbf{T}^{\#}(\mathbf{u} - \mathbf{u}_0) = \mathbf{0} \quad . \tag{A-43}$$

Hence $(\mathbf{u} - \mathbf{u}_0)$ is a solution of the homogeneous adjoint equation.

By assumption, then, $(\mathbf{u} - \mathbf{u}_0)$ is orthogonal to $\mathbf{u}$, so that

$$\begin{aligned} \langle \mathbf{u} - \mathbf{u}_0, \mathbf{u} \rangle &= 0 \\ &= \langle \mathbf{u} - \mathbf{u}_0, \mathbf{u} - \mathbf{u}_0 \rangle + \langle \mathbf{u} - \mathbf{u}_0, \mathbf{u}_0 \rangle \quad . \end{aligned} \tag{A-44}$$

However, $\mathbf{u}_0$ is in the range of $\mathbf{T}$, being the projection of $\mathbf{u}$ on the range. Also, $(\mathbf{u} - \mathbf{u}_0)$ is, by the definition of a projection, orthogonal to all vectors in the range of $\mathbf{T}$. Hence, the second term above vanishes, leaving only

$$\langle \mathbf{u} - \mathbf{u}_0, \mathbf{u} - \mathbf{u}_0 \rangle = 0 \qquad \text{(A-45)}$$

so that

$$\mathbf{u} = \mathbf{u}_0 \quad . \qquad \text{(A-46)}$$

Therefore $\mathbf{u}$ is, in fact, in the range of $\mathbf{T}$.

We have proven, then, both the sufficiency and the necessity of the condition.

The theorem that we originally wanted to prove, Theorem 2, is now simply a restatement of Theorem 4, for the space spanned by $\mathbf{w}$ is the null space of $\mathbf{T}^{\#}$ and the condition that $\mathbf{u}$ is orthogonal to all $\mathbf{w}$ defines the orthogonal complement. Hence, the range of $\mathbf{T}$ is the orthogonal complement of the null space of $\mathbf{T}^{\#}$.

We emphasize that our entire development does depend on the positive definiteness of the inner product—*i.e.*, that it be "proper." With this limitation, however, it does not depend on the particular inner product being used. As the inner product is changed, the adjoint operator changes but so does the definition of orthogonality, and hence of the orthogonal complement. The two changes complement each other in such a way that the range of $\mathbf{T}$ as the orthogonal complement of the null space of the adjoint of $\mathbf{T}$ is unaffected.

This, then, gives us the means of solving Eq. (A-1) in the general case.

We have that $\mathbf{u}$, which is known to be in the range of $\mathbf{T}$, is in the orthogonal complement of the null space of $\mathbf{T}^{\#}$. Hence, we can premultiply by Eq. (A-1) by $\mathbf{T}^{\#}$ and find

$$\mathbf{w} = \mathbf{T}^{\#}\mathbf{u} = \mathbf{T}^{\#}\mathbf{T}\mathbf{v} \qquad \text{(A-47)}$$

and know that $\mathbf{w}$ is not the null-vector if $\mathbf{u}$ is not.

Now the operator $\mathbf{T}^\#\mathbf{T}$ is self-adjoint under an inner product relation that is assumed to be proper:

$$(\mathbf{T}^\#\mathbf{T})^\# = \mathbf{T}^\#\mathbf{T} \quad . \tag{A-48}$$

Therefore, it is of simple structure, and the results of the preceding section apply. That is, a particular solution to Eq. (A-47) is obtained by considering $\Phi(\mathbf{T}^\#\mathbf{T})$, the minimum polynomial of $\mathbf{T}^\#\mathbf{T}$, and forming $\psi(\mathbf{T}^\#\mathbf{T})$ from it. This we can do since, firstly, $\mathbf{T}^\#\mathbf{T}$ is singular if $\mathbf{T}$ is, so that $\Phi(\mathbf{T}^\#\mathbf{T})$ has no constant term, and secondly, $\mathbf{T}^\#\mathbf{T}$ being of simple structure, the coefficient of the term in the first power of $\mathbf{T}^\#\mathbf{T}$ is not zero. Then we have that

$$\begin{aligned} \mathbf{v}_1 &= \psi(\mathbf{T}^\#\mathbf{T}) \\ &= \psi(\mathbf{T}^\#\mathbf{T})\mathbf{T}^\#\mathbf{u} \end{aligned} \tag{A-49}$$

and the general solution is

$$\mathbf{v} = \psi(\mathbf{T}^\#\mathbf{T})\mathbf{T}^\#\mathbf{u} + \mathbf{v}_0 \tag{A-50}$$

where $\mathbf{v}_0$ is any vector such that

$$\mathbf{T}\mathbf{v}_0 = 0 \quad . \tag{A-51}$$

This, then, gives us a general solution to our problem.

4. COMPARISON OF SOLUTIONS

The particular solution obtained in Eq. (A-49) is more involved than that of Eq. (A-18). Equation (A-49) is, of course, more general, in that it admits a $\mathbf{T}$ of any structure. But suppose $\mathbf{T}$ is of simple structure. We may ask if there are, in this case, problems in which it might still be worthwhile to use the formally more complicated solution of Eq. (A-49).

The two particular solutions are not, in general, the same. By Eq. (A-18), we find the particular solution that is in the range of $\mathbf{T}$. In Eq. (A-49) we find the particular solution that is in the range of $\mathbf{T}^\#$, and which therefore is in the orthogonal complement of the null space of $\mathbf{T}$. These two solutions will be the same only if the operator is normal with respect to the inner product relation being used. If the operator is not normal, the two solutions will differ by a vector that is in the null space of **T**.

The formally more complex solution of Eq. (A-49) depends upon the particular choice of the inner product relation. We can suggest then that it may be useful, even when Eq. (A-18) is usable, precisely because it permits us to choose the inner product relation that is most appropriate for the particular problem. It has, in other words, a built-in flexibility that may be of value.

APPENDIX B

PROPERTIES OF THE COMMUTATOR OPERATOR

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In this appendix we shall discuss some of the properties of the commutator operator. That is, we shall consider the equation

$$\mathbf{U} = \mathbf{T}_v\mathbf{W} = -T_w\mathbf{V} = [\mathbf{V},\mathbf{W}]$$

$$= (\mathbf{VW} - \mathbf{WV}) \qquad \text{(B-1)}$$

so that

$$\mathbf{T}_v = [\mathbf{V},\ ]$$

$$\mathbf{T}_w = [\mathbf{W},\ ] \quad . \qquad \text{(B-2)}$$

(Note that, for convenience, we are here dropping the scalar factor j that we usually include. It may be considered as being included in one of the matrices.)

Such equations arise in a number of different contexts in coupled mode theory, mechanics, quantum mechanics, and other fields.

In coupled mode theory, it arises naturally in the study of the noise power representations. If, for example, we have the equation

$$\frac{d\mathbf{x}}{dz} = -j\mathbf{R}\mathbf{x} \qquad \text{(B-3)}$$

where $\mathbf{R}$ is known to be K-hermitian for some constant, non-singular, hermitian $\mathbf{K}$, and $\mathbf{x}$ is a vector represented as a column matrix, we can, if we like, study the behavior of the dyad

$$\mathbf{W} = \mathbf{x}\mathbf{x}^{\dagger}\mathbf{K} \quad . \qquad \text{(B-4)}$$

Then we see that

$$\begin{aligned}\frac{d\mathbf{W}}{dz} &= \frac{d\mathbf{x}}{dz}\mathbf{x}^{\dagger}\mathbf{K} + \mathbf{x}\left(\frac{d\mathbf{x}}{dz}\right)^{\dagger}\mathbf{K} \\ &= -j\mathbf{R}\mathbf{x}\mathbf{x}^{\dagger}\mathbf{K} + j\mathbf{x}\mathbf{x}^{\dagger}\mathbf{R}^{\dagger}\mathbf{K} \\ &= -j\mathbf{R}\mathbf{x}\mathbf{x}^{\dagger}\mathbf{K} + j\mathbf{x}\mathbf{x}^{\dagger}\mathbf{K}\mathbf{R} \\ &= -j\mathbf{R}\mathbf{W} + j\mathbf{W}\mathbf{R} \\ &= -j\,[\mathbf{R},\mathbf{W}] = -j\mathbf{T}_R\mathbf{W} \quad . \end{aligned} \tag{B-5}$$

For the study of noise behavior in a system, $\mathbf{W}$ is a more convenient variable since its terms are spectral power densities, and thus are meaningful even when the components of $\mathbf{x}$ are "pathological" functions of time.

In addition, we have also observed that the general case, where $\mathbf{R}$ in Eq. (B-3) may be any matrix function of z, is extremely complicated. However, the class where $\mathbf{R}(z)$ is not unlimited but constrained to be such that its derivative is expressible as a commutator with some matrix $\mathbf{U}$,

$$\frac{d\mathbf{R}}{dz} = j\,[\mathbf{R},\mathbf{U}] = j\mathbf{T}_R\mathbf{U} \tag{B-6}$$

is an important class of systems, with some useful simplicities. It includes such non-uniform systems as the exponentially tapered transmission line, the distributed parametric amplifier, and the like.

A third aspect of the use of commutator operators is in the expansion of certain forms. In particular, we shall see that the form

$$e^{\mathbf{A}}\mathbf{B}e^{-\mathbf{A}}$$

is conveniently expandable in terms of commutators.

Our purpose here, then, is to establish the fundamental properties of these operators so as to provide the basis for their employment in various problems of analysis.

1. THE LINEAR VECTOR SPACE

We consider Eq. (B-1):

$$\mathbf{U} = \mathbf{T}_v \mathbf{W} = [\mathbf{V}, \mathbf{W}] \tag{B-7}$$

where we regard $\mathbf{T}_v$ as the operator in question.

The operator $\mathbf{T}_v$ operates on any $n \times n$ matrix, $\mathbf{W}$, to generate an $n \times n$ matrix $\mathbf{U}$.

We assert that the space of all $n \times n$ matrices is a linear vector space in which $\mathbf{T}_v$ is a linear homogeneous operator.

It may appear strange to call a square matrix a vector. However the properties that the elements of a set must have for the set to be a linear vector space do not depend on the form of the representation. We have merely adopted as a convention that is usually convenient the form of a column matrix. This is usually convenient, but not always, and is not necessary.

For a set to be a linear vector space, we require (1) that the operations of addition of two members and of multiplication by a scalar be defined, with the usual properties of addition and scalar multiplication, (2) that the sum of any two members of the set be in the set, and (3) that the product of any scalar times a member of the set is in the set.

The set of all $n \times n$ matrices with elements in a given field have these properties under the usual matrix addition, and scalar multiplication, and therefore form a linear vector space.

We could, if we liked, represent such a matrix in the standard form. We could, for example, form a column vector of dimensionality n^2 in which the components of the matrix were listed in some prescribed order. This is rarely worth doing, but does illustrate the possibility.

Of greater interest is the possibility of expanding an arbitrary member of the set in terms of some convenient complete set of standard matrices. For example, if $n = 2$, so that we are considering the space of 2×2 matrices, we can use the identity and the Pauli spin matrices used in Sec. II-B-4:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad . \tag{B-8}$$

Then any matrix, $\mathbf{A}$, can be expressed as

$$\mathbf{A} = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 + \alpha_4\sigma_4 \tag{B-9}$$

and we can represent $\mathbf{A}$ as col$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

More generally, it is sometimes convenient to use the set of K-dyads formed from some complete set of K-orthogonal maximally normalized vectors.

These various representations are possible and are sometimes convenient. However, it is not necessary to change to such a representation. For the immediate purpose, they merely serve to illustrate the fact that the set of all $n \times n$ matrices is a linear vector space—a fact which then is independent of the particular representation being used. We shall, in this section, use the representation as square matrices.

2. LINEARITY AND HOMOGENEITY OF THE COMMUTATOR

We have asserted that the operator, $\mathbf{T}_v$, is linear and homogeneous. It is linear because

$$\begin{aligned} \mathbf{T}_v(\alpha\mathbf{X} + \beta\mathbf{Y}) &= \mathbf{V}(\alpha\mathbf{X} + \beta\mathbf{Y}) - (\alpha\mathbf{X} + \beta\mathbf{Y})\mathbf{V} \\ &= \alpha(\mathbf{VX} - \mathbf{XV}) + \beta(\mathbf{VY} - \mathbf{YV}) \\ &= \alpha\mathbf{T}_v\mathbf{X} + \beta\mathbf{T}_v\mathbf{Y} \quad . \end{aligned} \tag{B-10}$$

It is homogeneous since, if $\mathbf{X}$ is the null matrix, then $\mathbf{T}_v\mathbf{X}$ is the null matrix.

It may be noted that this implies that, if $\mathbf{X}$ be represented in some manner as a column vector, as discussed in the last section, then the appropriate representation of $\mathbf{T}_v$ is a matrix.

If, for example, we are considering 2×2 matrices and use the representation of Eq. (B-9) in terms of the Pauli matrices, then it is easy to see that

$$\mathbf{X} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \frac{1}{2}\begin{pmatrix} B + C \\ -j(B - C) \\ A - D \\ A + D \end{pmatrix} . \tag{B-11}$$

If

$$\mathbf{V} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{B-12}$$

then we can find that, in this representation

$$\mathbf{T}_v\mathbf{X} = [\mathbf{V},\mathbf{X}] = \begin{pmatrix} 0 & j(\alpha - \delta) & -(\beta - \gamma) & 0 \\ -j(\alpha - \delta) & 0 & j(\beta + \gamma) & 0 \\ (\beta - \gamma) & -j(\beta + \gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{X} . \tag{B-13}$$

Again, we cite this only as an example.

We may note that the matrix of Eq. (B-13) is of rank 2. It has the zero eigenvectors

$$\mathbf{X}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{X}_2 = \begin{pmatrix} \beta + \gamma \\ -j(\beta - \gamma) \\ \alpha - \delta \\ 0 \end{pmatrix} . \tag{B-14}$$

This we would expect since, in the original representation of $\mathbf{T}_v$ as the commutator, $\mathbf{X} = \mathbf{I}$ and $\mathbf{X} = \mathbf{V}$ both commute with V, so that they are annihilated by the commutation operator.

We observe, in fact, that the commutator operator is necessarily singular. The subspace spanned by $\mathbf{I}$, $\mathbf{V}$, $\mathbf{V}^2$, ... is always annihilated by $\mathbf{T}_v$. More generally the null space of $\mathbf{T}_v$ is the space of matrices that commute with $\mathbf{V}$, and this is always a proper subspace.

As a final remark, we will add that the cross product of the vectors in vector analysis can be set up as a commutator relation. That is, if we have

$$\vec{u} = \vec{v} \times \vec{w} \tag{B-15}$$

it is possible to choose a representation of the vectors as matrices so that Eq. (B-15) takes the form of Eq. (B-7). One way, for example, is by using the isomorphism of Eq. (B-11) in reverse, after reducing the dimensionality to 3 by specifying, for example, that $\alpha_4 = 0$, or $A = -D$. Thus, Eq. (B-7) can be considered as a generalization of Eq. (B-15). The singularity of $\mathbf{T}_v$ is, then, a reflection of the fact that the cross product of Eq. (B-15) is singular, and vanishes if $\vec{v} \propto \vec{w}$.

3. INNER PRODUCT RELATION

We now wish to obtain an inner product relation what will metricize our space of $n \times n$ matrices. For this purpose, we need a procedure whereby we can obtain from two elements, $\mathbf{U}$ and $\mathbf{V}$ of the space, a scalar valued function that is linear in $\mathbf{U}$ and $\mathbf{V}$. Specifically, we need a scalar valued function which we will write as $\langle \mathbf{U},\mathbf{V} \rangle$, such that

$$\langle \mathbf{U},\alpha\mathbf{V} \rangle = \alpha\langle \mathbf{U},\mathbf{V} \rangle \tag{B-16}$$

$$\langle \mathbf{V}\ \mathbf{U} \rangle = \langle \mathbf{U}\ \mathbf{V} \rangle^* \tag{B-17}$$

$$\langle \mathbf{U},(\mathbf{V}_1 + \mathbf{V}_2) \rangle = \langle \mathbf{U},\mathbf{V}_1 \rangle + \langle \mathbf{U},\mathbf{V}_2 \rangle \quad . \tag{B-18}$$

We also want this relation to be positive definite so that the inner product relation is "proper" and we can apply the results of Appendix A. That is, we want

$$\langle \mathbf{U},\mathbf{U} \rangle \geq 0 \tag{B-19}$$

with the equality holding if and only if $\mathbf{U}$ is the null matrix.

Our first problem is how to obtain a scalar valued function from matrices. The scalars that are naturally associated with a matrix are the sums of the K-rowed principal minors, which are the coefficients of the characteristic equation and which also are the symmetrical elementary functions of the eigenvalues.

Of these scalars, only the first, which is the trace, or the sum of the diagonal terms, is linear in the sense that the trace of a scalar, α, times a matrix, $\mathbf{A}$, is α times the trace of $\mathbf{A}$.

So it is reasonable to expect that we will want to use the trace of some combination of $\mathbf{U}$ and $\mathbf{V}$.

One combination that will work is the product of $\mathbf{U}^\dagger$ and $\mathbf{U}$

$$<\mathbf{U},\mathbf{V}> = \mathrm{tr}\, \mathbf{U}^\dagger\mathbf{V}.$$

This evidently obeys Eqs. (B-16), (B-17), and (B-18). That it also is positive definite can be seen by noting that, if U_{ij} is the ij component of $\mathbf{U}$, and V_{ij} of $\mathbf{V}$, then

$$\begin{aligned} <\mathbf{U},\mathbf{V}> &= \mathrm{tr}\, \Sigma\, U^*_{ji}V_{jk} \\ &= \sum_{ij} U^*_{ji}V_{ji} \qquad \text{(B-20)} \end{aligned}$$

and

$$<\mathbf{U}\,\mathbf{U}> = \Sigma\, |U_{ji}|^2 \geq 0 \quad . \qquad \text{(B-21)}$$

Other inner product relations, also based on the trace of the product of $\mathbf{U}^\dagger$ and $\mathbf{V}$, but with appropriate other factors, can also be found. However, it is not apparent that these are useful, and we shall not go into them here.

We will note for future reference the important property of the trace that

$$\mathrm{tr}\, \mathbf{AB} = \mathrm{tr}\, \mathbf{BA} \quad . \qquad \text{(B-22)}$$

This follows since

$$\begin{aligned} \mathrm{tr}\, \mathbf{AB} &= \mathrm{tr} \sum_j A_{ij}B_{jk} \\ &= \sum_{ij} A_{ij}B_{ji} \\ &= \sum_{ij} B_{ij}A_{ji} = \mathrm{tr}\, \mathbf{BA} \quad . \qquad \text{(B-23)} \end{aligned}$$

It is this property that is involved in Eq. (B-17). However, it should also be noted that it is *not* true that the trace of the product of any number of matrices is independent of the order. We do have, by repeated application of this theorem, that

$$\mathrm{tr}\,\mathbf{ABC} = \mathrm{tr}\,\mathbf{CAB} = \mathrm{tr}\,\mathbf{BCA}$$

but this is not, in general, equal to tr $\mathbf{CBA}$, for example. The trace of the product of a number of factors is independent of a cyclic permutation of the factors, but not, in general, of any permutation.

4. THE ADJOINT OPERATOR

Our principal interest in obtaining an inner product relation is to give us the adjoint operator.

We will show that, under the inner product relation of Sec. 3 above,

$$\mathbf{T}_v^{\#} = \mathbf{T}_{v^\dagger} = [\mathbf{V}^\dagger,] \tag{B-24}$$

is the adjoint operator, $\mathbf{T}_v^{\#}$, of

$$\mathbf{T}_v = [\mathbf{V},] \tag{B-25}$$

This follows since

$$\begin{aligned} \langle \mathbf{U}, \mathbf{T}_v \mathbf{W} \rangle &= \mathrm{tr}\,\{\mathbf{U}^\dagger(\mathbf{VW} - \mathbf{WV})\} \\ &= \mathrm{tr}\,\mathbf{U}^\dagger\mathbf{VW} - \mathrm{tr}\,\mathbf{U}^\dagger\mathbf{WV} \\ &= \mathrm{tr}\,\mathbf{U}^\dagger\mathbf{VW} - \mathrm{tr}\,\mathbf{VU}^\dagger\mathbf{W} \end{aligned} \tag{B-26}$$

The second term in the last line is a cyclic permutation of the term above it, and so does not change the trace. It follows then, that

$$\begin{aligned} \langle \mathbf{U}, \mathbf{T}_v \mathbf{W} \rangle &= \mathrm{tr}\,\{(\mathbf{U}^\dagger\mathbf{V} - \mathbf{VU}^\dagger)\mathbf{W}\} \\ &= \mathrm{tr}\,\{(\mathbf{V}^\dagger\mathbf{U} - \mathbf{UV}^\dagger)^\dagger\mathbf{W}\} \\ &= \langle \mathbf{T}_{v^\dagger}\mathbf{U}, \mathbf{W} \rangle \quad . \end{aligned} \tag{B-27}$$

This gives us, then, an adjoint operator under a proper inner product.

With this adjoint operator the results of Appendix A, are immediately applicable. For example, according to Theorem 4, Appendix A, we know that Eq. (B-1) is soluble for $\mathbf{W}$ if and only if $\mathbf{U}$ is orthogonal to every matrix that commutes with $\mathbf{V}^\dagger$.

In particular, if we consider the equation

$$\frac{d\mathbf{R}}{dz} = j[\mathbf{R},\mathbf{A}] = j\mathbf{T}_R\mathbf{A} \tag{B-28}$$

we find that it is soluble for $\mathbf{A}$ if and only if $\mathbf{R}' = d\mathbf{R}/dz$ is orthogonal to each of the sequence $\mathbf{R}^{\dagger 0} = I$, $\mathbf{R}^\dagger$, $\mathbf{R}^{\dagger 2}$, ... Hence

$$\begin{aligned}
<\mathbf{I},\mathbf{R}'> &= \operatorname{tr} \mathbf{R}' = 0 \\
<\mathbf{R}^\dagger,\mathbf{R}'> &= \operatorname{tr} (\mathbf{R}\mathbf{R}') = 0 \\
<\mathbf{R}^{\dagger 2},\mathbf{R}'> &= \operatorname{tr} (\mathbf{R}^2\mathbf{R}') = 0 \\
&\text{etc.}
\end{aligned} \tag{B-29}$$

Since, for example, from Eq. (B-23)

$$\begin{aligned}
\operatorname{tr} (\mathbf{R}\mathbf{R}') &= \operatorname{tr} (\mathbf{R}'\mathbf{R}) = \frac{1}{2}\operatorname{tr} (\mathbf{R}\mathbf{R}' + \mathbf{R}'\mathbf{R}) \\
&= \frac{1}{2}\{\operatorname{tr} (\mathbf{R}^2)'\}
\end{aligned} \tag{B-30}$$

it follows that the derivatives of all the powers of $\mathbf{R}$ must have zero trace. It follows directly that the eigenvalues of $\mathbf{R}$ must be constant.

Given, then, that $\mathbf{R}$ has constant eigenvalues so that Eq. (B-28) is soluble, we can, then, solve Eq. (B-28) for $\mathbf{A}$ by the methods of the last section. If $\mathbf{T}_R$ is an operator of simple structure, or at least has no chain of generalized eigenvectors of zero eigenvalue then there exists a polynomial function of $\mathbf{T}_R$, $\psi(\mathbf{T}_R)$ such that

$$\mathbf{A} = \psi(\mathbf{T}_R)\mathbf{R}' + \mathbf{A}_0$$

where $\mathbf{A}_0$ is a matrix in the null space of $\mathbf{T}_R$.

It should be noted that $\mathbf{T}_R^2$, for example, is the result of applying $\mathbf{T}_R$ twice. Thus it is

$$\mathbf{T}_R^2\mathbf{A} = [\mathbf{R}, \mathbf{T}_R\mathbf{A}]$$

$$= [\mathbf{R}(\mathbf{R},\mathbf{A})] = [_2\mathbf{R},\mathbf{A}] \qquad \text{(B-31)}$$

where the symbol $[_2]$ means the second commutator.

The power of any operator $\mathbf{T}$ does not mean the power of a matrix, unless the operator is matrix multiplication by a matrix. It means, instead, the repeated application of the operator.

We shall not pursue the development of Eq. (B-28) further at the moment. We shall only note that, for $\mathbf{R}$ to be such as to permit its variation to be described by Eq. (B-28), it is necessary and sufficient for the trace of $\mathbf{R}^k (k = 1, 2, \ldots)$ to be constant.

5. EIGENVECTORS AND EIGENVALUES, V OF SIMPLE STRUCTURE

We have found that the operator $\mathbf{T}_v$ is linear and homogeneous. Therefore, it has at least one eigenvector. That is, there is at least one $\mathbf{U}_{ij}$, and one μ_{ij} such that

$$\mathbf{T}_v\mathbf{U}_{ij} = \mu_{ij}\mathbf{U}_{ij} \quad . \qquad \text{(B-32)}$$

(The reason for using the double index will become apparent shortly.)

The fact that the $\mathbf{U}_{ij}$ are matrices does not affect their being eigenvectors in the space of $n \times n$ matrices.

If we use the dyad expansion on the eigenvectors of $\mathbf{V}$, $\mathbf{V}$ being assumed to be of simple structure then

$$\mathbf{V} = \sum_i \sigma_i \lambda_i \mathbf{v}_i \mathbf{v}^\dagger_{\sim i} \mathbf{K}$$

if $\mathbf{K}$ is a metric for which $\mathbf{V}$ is at least K-normal. The set $\{\mathbf{v}_i\}$ are K-orthogonal and maximally normalized.

Then

$$\mathbf{E}_{ij} = \sigma_i \mathbf{v}_i \mathbf{v}^{\dagger}_{\sim j}\mathbf{K} \tag{B-33}$$

is an eigenvector of $\mathbf{T}_v$ with eigenvalue $(\lambda_i - \lambda_j)$, for

$$\begin{aligned}
\mathbf{T}_v\mathbf{E}_{ij} &= \sum_k \{\sigma_k\lambda_k\mathbf{v}_k\mathbf{v}^{\dagger}_{\sim k}\mathbf{K}\sigma_i\mathbf{v}_i\mathbf{v}^{\dagger}_{\sim j}\mathbf{K} - \sigma_i\mathbf{v}_i\mathbf{v}^{\dagger}_{\sim j}\mathbf{K}\sigma_k\lambda_k\mathbf{v}_k\mathbf{v}^{\dagger}_{\sim k}\mathbf{K}\} \\
&= \lambda_i\sigma_i\mathbf{v}_i\mathbf{v}^{\dagger}_{\sim j}\mathbf{K} - \lambda_j\sigma_i\mathbf{v}_i\mathbf{v}^{\dagger}_{\sim j}\mathbf{K} \\
&= (\lambda_i - \lambda_j)\mathbf{E}_{ij} \quad .
\end{aligned} \tag{B-34}$$

Further, since the set $\{\mathbf{E}_{ij}\}$ is complete, these are the entire set of eigenvectors of $\mathbf{T}_v$, and $\mathbf{T}_v$ is of simple structure if $\mathbf{V}$ is.

The null space of $\mathbf{T}_v$ is the space spanned by the set $\{\mathbf{E}_{ii}\}$, plus any additional terms due to degeneracies.

The minimum polynomial of $\mathbf{T}_v$ is the product

$$\Pi(\mathbf{T}_v - \mu_{ij}\mathbf{I}) \tag{B-35}$$

over all terms with distinct μ_{ij}. However, for each μ_{ij} we evidently have

$$\mu_{ji} = \lambda_j - \lambda_i = -\mu_{ij} \quad . \tag{B-36}$$

The minimum polynomial contains the single factor $\mathbf{T}_v$, corresponding to the zero eigenvalue. The other factors then occur in pairs:

$$(\mathbf{T}_v - \mu_{ij}\mathbf{I})(\mathbf{T}_v - \mu_{ji}\mathbf{I}) = \mathbf{T}_v^2 - (\mu_{ij})^2\mathbf{I} \quad . \tag{B-37}$$

Hence the minimum polynomial contains only odd powers of $\mathbf{T}_v$, and can be written as

$$\begin{aligned}
\phi(\mathbf{T}_v) &= \{a_1 + (a_3 + a_5\mathbf{T}^2 + \ldots)\mathbf{T}^2\}\mathbf{T} \\
&= a_1[1 - \theta(\mathbf{T})\mathbf{T}^2]\mathbf{T}
\end{aligned} \tag{B-38}$$

where

$$\theta(\mathbf{T}) = -\frac{1}{a_1}(a_3 + a_5\mathbf{T}^2 + \ldots) \quad . \tag{B-39}$$

As discussed in Sec. II-C-1, a_1 does not vanish providing $\mathbf{T}_v$ is of simple structure, as we have shown it is if $\mathbf{V}$ is of simple structure. Hence, $\theta(\mathbf{T})$ exists and Eq. (B-38) is valid. In other words, if $\mathbf{V}$ is of simple structure, we can always find a polynomial function of $\mathbf{T}_v$, $\theta(\mathbf{T}_v)$, such that, for any $\mathbf{U}$

$$\mathbf{T}_v\mathbf{U} = \theta(\mathbf{T}_v)\mathbf{T}^3\mathbf{U} \quad .$$

We shall not discuss the more general case where $\mathbf{V}$ may not be of simple structure. Essentially the same results apply except that if $\mathbf{V}$ is not of simple structure, then $\mathbf{T}_v$ is not either, and, in fact, must then have a chain of generalized eigenvectors with zero eigenvalues. Therefore, we must always use the more general procedure for its "inversion" as discussed in Appendix A.

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Regional Offices and Laboratories

Southern California Laboratories
820 Mission Street
South Pasadena, California

Washington Office
808 17th Street, N.W.
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Representatives

Honolulu, Hawaii
1125 Ala Moana Blvd.
Honolulu, Hawaii

London, England
19, Upper Brook Street
London, W. 1, England

Milan, Italy
Via Macedonio Melloni, 49
Milano, Italy

Toronto, Ontario, Canada
Room 710, 67 Yonge St.
Toronto, Ontario, Canada